ON THE VARIETY OF INVARIANT SUBSPACES OF
A FINITE-DIMENSIONAL LINEAR OPERATOR

BY

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ABSTRACT. If V is a finite-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$ and $A \in \operatorname{Hom}(V)$, the set $S_A(k)$ of k-dimensional A-invariant subspaces is a compact subvariety of the Grassmann manifold $G^k(V)$, but it need not be a Schubert variety. We study the topology of $S_A(k)$. We reduce to the case where A is nilpotent. In this case we prove that $S_A(k)$ is connected but need not be a manifold. However, the subset of $S_A(k)$ consisting of those subspaces with a fixed cyclic structure is a regular submanifold of $G^k(V)$.

1. Introduction. If V is a finite-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$ and $A \in \operatorname{Hom}(V)$, the set $S_A(k)$ of k-dimensional A-invariant subspaces is a compact subvariety of $G^k(V)$, the Grassmann manifold of k-dimensional subspaces of V. We study the topological structure of $S_A(k)$. We show that it suffices to consider the case where A is nilpotent. In this case, we prove that $S_A(k)$ is always connected (Theorem 6). However, if A is neither cyclic nor semisimple, $S_A(k)$ need not be a manifold. We show this by completely characterizing the geometric structure of $S_A(k)$ for the class of nilpotent operators whose Jordan canonical form consists of one block of arbitrary size followed by an arbitrary number of 1×1 blocks (Theorem 5). For such an operator, $S_A(k)$ is a union of Grassmann manifolds which intersect each other along Grassmannian submanifolds.

This motivates us to consider the subset of $S_A(k)$ consisting of all k-dimensional A-invariant subspaces with a given cyclic structure. We prove that every such subset is a regular submanifold of $G^k(V)$ (Theorem 7). Our proof of this result is interesting because we construct the submanifold as an orbit space under the action of a special subgroup of $Gl(k)$. The charts we construct also give convenient parametrizations of each of these submanifolds.

In the final section of this paper, we consider the question of whether $S_A(k)$ is always expressible as a union or intersection of Schubert varieties. By considering an example in some detail, we show that this question has a negative answer. Thus, the varieties of invariant subspaces of finite-dimensional linear operators form an interesting and, as far as we are aware, unstudied class of subvarieties of the Grassmann manifold.

The variety S_A has an important application in the field of automatic control. A crucial equation in linear-quadratic control theory is the algebraic Riccati equation $-A'K - KA + KBB'K - Q = 0$, where A, B, and Q are real matrices of dimensions $n \times n$, $n \times m$, and $n \times n$ respectively, and $Q = Q'$. (Prime denotes transpose.) K is an unknown real symmetric matrix. J. C. Willems [8] proved that under mild assumptions there is an $n \times n$ matrix A^+ which can be constructed from the coefficient matrices A, B, Q such that S_{A^+} is in bijection with the solution set of the equation. We have proved elsewhere [7] that this bijection is actually a homeomorphism. Hence, the topological properties of the solution set of the algebraic Riccati equation are the same as those of S_{A^+}. This was our original motivation for exploring the subject of this paper.

Specialized notation will be defined when it is initially used. Unless otherwise stated, all morphisms are to be interpreted as morphisms of algebraic varieties. We use the symbol " $\approx$ " to indicate an isomorphism of algebraic varieties.

2. Preliminaries. Let V be an n-dimensional vector space over the field $\mathscr{F}$ of real or complex numbers. $\mathrm{Hom}(V)$ is the space of linear mappings of V into itself. Let A be a fixed element of $\mathrm{Hom}(V)$. A linear subspace $S \subseteq V$ is A-invariant if $A(S) \subseteq S$. Let S_A be the set of A-invariant subspaces of V, and let $S_A(k)$ be the subset of S_A consisting of all k-dimensional A-invariant subspaces.

Let $G^k(V)$ be the Grassmann manifold of k-dimensional (linear) subspaces of V. Form the disjoint union $\bigsqcup_{k=0}^{n} G^k(V)$ and give it the topology generated by the open sets in $G^0(V), \ldots, G^n(V)$. Since $S_A \subseteq \bigsqcup_{k=0}^{n} G^k(V)$, S_A inherits a topology from the disjoint union. Since $S_A(k) \subseteq G^k(V)$, it follows that if $S_1, S_2 \in S_A$ and $\dim S_1 \neq \dim S_2$, then S_1 and S_2 belong to different connected components of S_A. Consequently, to understand the topological structure of S_A, it suffices to understand the structure of each subset $S_A(k)$, $k = 0, 1, \ldots, n$.

REMARK 1. If $\lambda \in \mathscr{F}$, then $S_{A+\lambda I} = S_A$.

THEOREM 1. $S_A(k)$ *is a compact subvariety of* $G^k(V)$.

PROOF. By Remark 1 we can assume that A is nonsingular. Define a mapping ϕ_A: $G^k(V) \to G^k(V)$ by $\phi_A(S) \equiv A(S)$. Then ϕ_A is a regular mapping and $S_A(k)$ is its fixed point set. It follows that $S_A(k)$ is a subvariety of $G^k(V)$ [5, p. 59]. Since $G^k(V)$ is compact, the same is true of $S_A(k)$. $\square$

COROLLARY. S_A *is compact.*

$G^k(V)$ is a projective algebraic variety by the classical Plücker embedding [4, p. 209]. Let $\Lambda^k(V)$ be the kth exterior power of V. A nonsingular linear transformation $A \in \mathrm{Hom}(V)$ induces a nonsingular linear transformation $\Lambda^k(A) \in \mathrm{Hom}(\Lambda^k(V))$. Let $\mathbf{P}(\Lambda^k(V))$ denote the projective space of lines through the origin in $\Lambda^k(V)$. If $X \in \Lambda^k(V) - \{0\}$, let $[X]$ denote the corresponding element of $\mathbf{P}(\Lambda^k(V))$. Let $\{v_1, \ldots, v_k\}$ be a set of k linearly independent vectors in V. The Plücker map p: $G^k(V) \to \mathbf{P}(\Lambda^k(V))$ is defined by $p(\mathrm{Sp}\{v_1, \ldots, v_k\}) = [v_1 \wedge \cdots \wedge v_k]$.

Now, $\mathrm{Sp}\{v_1, \ldots, v_k\} = \mathrm{Sp}\{Av_1, \ldots, Av_k\}$ iff $v_1 \wedge \cdots \wedge v_k = \lambda Av_1 \wedge \cdots \wedge Av_k$ for some nonzero scalar λ. Thus $\mathrm{Sp}\{v_1, \ldots, v_k\}$ is A-invariant iff $[v_1 \wedge \cdots \wedge v_k]$ is

$\Lambda^k(A)$-invariant. Hence, the image of $S_A(k)$ under the Plücker map is the intersection of $p(G^k(V))$ with the *union* (not the sum) of the *eigenspaces* of $\Lambda^k(A)$. (We do not mean generalized eigenspaces here.) Thus, $p(S_A(k))$ is a disjoint union of projective varieties. Each such variety is defined by a set of homogeneous linear equations which define the corresponding eigenspace of $\Lambda^k(A)$, in addition to the set of homogeneous quadratic equations (the Plücker equations) which define $p(G^k(V))$. In particular, if A has only one distinct eigenvalue, then the same is true of $\Lambda^k(A)$, so in this case $p(S_A(k))$ is a single projective variety.

3. Reduction to the nilpotent case. In this section, we show that there is no loss of generality in assuming that A is nilpotent. Let $p(s)$ be the characteristic polynomial of A, and let $p(s) = p_1(s)^{m_1} \cdots p_q(s)^{m_q}$ be its prime factorization in the ring $\mathcal{F}[s]$. Let $E_i \equiv \ker p_i(A)^{m_i}$. Then the decomposition of V into its primary components with respect to A is $V = E_1 \oplus \cdots \oplus E_q$. Let n_i be the dimension of E_i. Then $n_i = m_i \cdot \deg p_i(s)$. Let A_i be the restriction of A to the A-invariant subspace E_i, and let $\pi_i \colon V \to E_i$ be the projection onto E_i along the direct sum of the remaining primary components.

The set of all invariant subspaces of A_i, S_{A_i}, is the topological disjoint union $\bigsqcup_{k_i=0}^{n_i} S_{A_i}(k_i)$, and $S_{A_i}(k_i)$ is a compact subvariety of $G^{k_i}(E_i)$. We have a natural bijection $\phi \colon S_A \to S_{A_1} \times \cdots \times S_{A_q}$ given by $\phi(S) = (\pi_1(S), \ldots, \pi_q(S))$.

THEOREM 2. *ϕ is an isomorphism of projective varieties.*

PROOF. First we show that ϕ^{-1} is a regular mapping. Since S_{A_i} is the topological disjoint union $\bigsqcup_{k_i=0}^{n_i} S_{A_i}(k_i)$, it suffices to consider the restriction of ϕ^{-1} to $S_{A_1}(k_1) \times \cdots \times S_{A_q}(k_q)$. Let $k = k_1 + \cdots + k_q$, let $S_i \in S_{A_i}(k_i)$ ($i = 1, \ldots, q$), and let $S = \phi^{-1}(S_1, \ldots, S_q) = S_1 \oplus \cdots \oplus S_q$. By choosing a basis, we can identify V with $\mathcal{F}^n$, and then we can choose an $n \times n_i$ full rank matrix B_i whose columns span the n_i-dimensional subspace E_i ($i = 1, \ldots, q$). Choose standard charts for $G^k(V)$ around S and for $G^{k_i}(E_i)$ around S_i ($i = 1, \ldots, q$). (The standard charts for the Grassmann manifold are described in §5.) Each chart for $G^k(V)$ associates a subspace with an $n \times k$ matrix which contains a $k \times k$ identity submatrix. The remaining $(n - k)k$ entries are the coordinates of the subspace. The standard charts for $G^{k_i}(E_i)$ are described similarly. Then in local coordinates, the mapping ϕ^{-1} consists of (1) assembling the $(n_i - k_i)k_i$ coordinates for an element of $G^{k_i}(E_i)$ together with a $k_i \times k_i$ identity matrix into an $n_i \times k_i$ matrix, say X_i ($i = 1, \ldots, q$); (2) forming the full rank $n \times k$ matrix $[B_1 X_1, \ldots, B_q X_q]$; (3) multiplying $[B_1 X_1, \ldots, B_q X_q]$ on the right by the inverse of the $k \times k$ submatrix corresponding to the chosen chart for $G^k(V)$ (thereby obtaining a $k \times k$ identity submatrix); (4) reading off the remaining $(n - k)k$ entries. This shows that the local expression for ϕ^{-1} consists of $(n - k)k$ regular functions, so ϕ^{-1} is a regular mapping.

Now we show that ϕ is a regular mapping. Since ϕ^{-1} is continuous, $S_{A_1} \times \cdots \times S_{A_q}$ is compact, and S_A is Hausdorff, it follows that ϕ is continuous. Since $S_{A_1} \times \cdots \times S_{A_q}$ is the topological disjoint union $\bigsqcup_{k_1} \cdots \bigsqcup_{k_q} S_{A_1}(k_1) \times \cdots \times S_{A_q}(k_q)$, it follows that S_A is the topological disjoint union of the corresponding inverse images. Consequently, to show that ϕ is regular, it suffices to examine the restriction of ϕ to

$\phi^{-1}(S_{A_1}(k_1) \times \cdots \times S_{A_q}(k_q))$. Restricted to this domain, the mapping $S \mapsto \pi_i(S)$ is a mapping of a (closed and open) subset of $S_A(k)$ onto $S_{A_i}(k_i)$. Choosing standard charts for $G^k(V)$ and $G^{k_i}(E_i)$, the local coordinate expression for this mapping consists of (1) assembling the $(n-k)k$ coordinates for an element of $G^k(V)$ together with a $k \times k$ identity matrix into an $n \times k$ matrix, say X; (2) multiplying X on the left by the $n \times n$ matrix P_i representing the projection π_i; (3) forming a full rank $n \times k_i$ submatrix, say Y_i, from the $n \times k$ rank k_i matrix $P_i X$; (4) multiplying on the left by the left inverse $\tilde{B}_i$ of the $n \times n_i$ basis matrix B_i, thus obtaining the $n_i \times k_i$ full rank matrix $\tilde{B}_i Y_i$; (5) multiplying $\tilde{B}_i Y_i$ on the right by the inverse of the $k_i \times k_i$ submatrix corresponding to the chosen chart for $G^{k_i}(E_i)$ (thereby obtaining a $k_i \times k_i$ identity submatrix); (6) reading off the remaining $(n_i - k_i)k_i$ entries. In step (3), it is clear that if a certain choice of k_i columns is made to obtain Y_i from $P_i X$, those same k_i columns can be chosen in some neighborhood. (I.e. the choice of k_i columns can be held fixed locally.) It follows immediately that the local expression for the mapping $S \mapsto \pi_i(S)$ consists of regular functions. Thus, ϕ is a biregular mapping. $\square$

By Theorem 2, it suffices to characterize S_A for the case where the characteristic polynomial $p(s)$ of A is a power of an irreducible polynomial in $\mathfrak{F}[s]$, say $p(s) = r(s)^m$. If $\mathfrak{F} = \mathbf{C}$, then $r(s)$ is linear, and A differs from a nilpotent operator by a multiple of the identity. Since $S_{A+\lambda I} = S_A$, there is no loss of generality in taking A to be nilpotent.

If $\mathfrak{F} = \mathbf{R}$, $r(s)$ may be either linear or quadratic. If $r(s)$ is linear, A can be replaced by a nilpotent operator with the same set of invariant subspaces. Therefore, suppose that $r(s)$ is quadratic. Let V^+ denote the complexification of the real vector space V, and let $A^+ \in \mathrm{Hom}(V^+)$ denote the complexification of A. A^+ has a pair of complex conjugates, say λ and $\bar{\lambda}$, as its distinct eigenvalues. If F is the primary component of V^+ corresponding to λ, then the conjugate subspace $\bar{F}$ is the primary component of V^+ corresponding to $\bar{\lambda}$.

Let A_F^+ be the restriction of A^+ to F. Let $\pi_F: V^+ \to F$ be the projection onto F along $\bar{F}$. We define a mapping $\psi: S_A \to S_{A_F^+}$ by $\psi(S) \equiv \pi_F(S^+)$, where S^+ is the complexification of the subspace S.

PROPOSITION 1. ψ is an isomorphism of real projective varieties.

PROOF. It is clear that ψ is a bijection. Since S_A is the topological disjoint union $\bigsqcup_{k=0}^m S_A(2k)$ and $S_{A_F^+}$ is the topological disjoint union $\bigsqcup_{k=0}^m S_{A_F^+}(k)$, it suffices to show that $\psi: S_A(2k) \to S_{A_F^+}(k)$ is a morphism of real projective varieties. $S_A(2k)$ is a subvariety of the real projective variety $G^{2k}(V)$ (where $\dim V = 2m$). $S_{A_F^+}(k)$ is a subvariety of the complex projective variety $G^k(F)$. However, the m-dimensional complex vector space F is also a $2m$-dimensional real vector space which we denote by F_r. Furthermore, every k-dimensional complex subspace M of F is also a $2k$-dimensional real subspace of F_r which we denote by M_r. Thus, the complex variety $G^k(F)$ is naturally identified with a subvariety of the real variety $G^{2k}(F_r)$. By this identification, $S_{A_F^+}(k)$ is a subvariety of $G^{2k}(F_r)$.

Each vector $z \in V^+$ is of the form $z = x + iy$ with $x, y \in V$. Following the obvious analogy, we denote x by $\text{Re}(z)$, the real part of the vector z. We have a natural mapping $\eta: F_r \to V$ given by $\eta(z) \equiv \text{Re}(z)$. η is clearly **R**-linear and, from the fact that $F \oplus \overline{F} = V^+$, it follows that η is bijective. Thus, η is a natural isomorphism of F_r onto V. It therefore induces an isomorphism (of real projective varieties) $\tilde{\eta}: G^{2k}(F_r) \to G^{2k}(V)$ defined by $\tilde{\eta}(M) \equiv \eta(M)$. Furthermore, the restriction of $\tilde{\eta}$ to $S_{A_F^+}(k)$ is precisely ψ^{-1}. To see this, let $S \in S_A(2k)$ and let $M \equiv \psi(S) \in S_{A_F^+}(k)$. Then by definition, $M = \pi_F(S^+)$. However, since π_F is the projection onto the primary component F, $\pi_F(S^+) = S^+ \cap F$. Suppose that $x \in \tilde{\eta}(M)$. Then $\exists y \in V$ such that $x + iy \in M$. Thus, $x + iy \in S^+$, which means that $x, y \in S$. Hence $\tilde{\eta}(M) \subseteq S$. However, $\dim \tilde{\eta}(M) = 2k = \dim S$, so $\tilde{\eta}(M) = S$, as required. Thus, ψ is the restriction to $S_A(2k)$ of a biregular mapping. $\square$

Combining Proposition 1 with Theorem 2 yields a refinement of Theorem 2 in the case where $\mathscr{F} = \mathbf{R}$.

THEOREM 3. *Let $\mathscr{F} = \mathbf{R}$. Let $E_1, \ldots, E_p$ be the primary components of A corresponding to the distinct real eigenvalues of A, and let $F_1, \overline{F}_1, \ldots, F_q, \overline{F}_q$ be the primary components of A^+ corresponding to its nonreal eigenvalues. Let A_i denote the restriction of A to E_i $(i = 1, \ldots, p)$, and let A_j^+ denote the restriction of A^+ to F_j $(j = 1, \ldots, q)$. Then S_A is isomorphic to the product $S_{A_1} \times \cdots \times S_{A_p} \times S_{A_1^+} \times \cdots \times S_{A_q^+}$ as real projective varieties.*

REMARK 2. Since A_i $(i = 1, \ldots, p)$ and A_j^+ $(j = 1, \ldots, q)$ differ from nilpotent operators by multiples of the identity, it follows from Theorem 3 that to characterize S_A for all real operators A, it suffices to characterize S_A for all real and all complex nilpotent operators A.

The results of this section show that to characterize the structure of S_A for every real and every complex A, it is sufficient to do the same for every real and every complex nilpotent A. Furthermore, it is trivial to show that if A and B are similar operators, then S_A and S_B are isomorphic. In fact, if $B = PAP^{-1}$, then $S_B = PS_A$, where we are identifying P with the mapping of subspaces which it induces. Consequently, it suffices to characterize the structure of S_A for the case where $V = \mathscr{F}^n$ ($\mathscr{F} = \mathbf{R}$ or $\mathbf{C}$) and A is a nilpotent matrix in lower Jordan canonical form. For the remainder of this paper, we shall consider such an A.

4. Topological structure of $S_A(k)$. Let A be an $n \times n$ nilpotent matrix in lower Jordan form. Let $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r \geqslant 1$ be the sizes of the blocks in A ($m_1 + \cdots + m_r = n$). Let $\{e_1, \ldots, e_n\}$ be the standard basis for $V \equiv \mathscr{F}^n$. Then $Ae_j = e_{j+1}$ for $j \neq m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_r$, and $Ae_j = 0$ for $j = m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_r$. Let

$$W_1 \equiv \text{Sp}\{e_1, \ldots, e_{m_1}\},$$

$$W_2 \equiv \text{Sp}\{e_{m_1+1}, \ldots, e_{m_1+m_2}\}, \ldots, W_r = \text{Sp}\{e_{m_1+\cdots+m_{r-1}+1}, \ldots, e_n\}.$$

Then W_i is an m_i-dimensional cyclic subspace (with respect to A) and $V = W_1 \oplus \cdots \oplus W_r$. We say that V has *cyclic structure* $(m_1, \ldots, m_r)$ and that $W_1 \oplus \cdots \oplus W_r$

is a cyclic decomposition of V. Although the cyclic structure of V is uniquely determined by A, there is no *natural* cyclic decomposition of V. In general, there are many cyclic decompositions of V (with respect to A). This is in contrast to the decomposition of a vector space into primary components (relative to a given operator). The primary component decomposition is natural (no choice is involved), and it is essentially this feature which enables us to reduce to the special case of nilpotent operator. Since the cyclic decomposition is not natural, we do not expect to be able to reduce further to the case of a nilpotent operator which is also cyclic.

Let $(c_1, \ldots, c_s)$ be the partition of n which is *conjugate* to $(m_1, \ldots, m_r)$. $(c_1, \ldots, c_s)$ is easily found by drawing the Ferrers diagram for $(m_1, \ldots, m_r)$ and reversing the roles of the rows and columns.

EXAMPLE 1. Let $(m_1, m_2, m_3) = (5, 3, 1)$. Then the corresponding Ferrers diagram is:
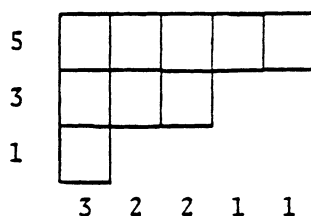
The column sums $(3, 2, 2, 1, 1)$ give the conjugate partition $(c_1, c_2, c_3, c_4, c_5)$.

Now, let $V_i \equiv \operatorname{Ker} A^i$, $i = 0, 1, \ldots, m_1$. Then $0 = V_0 \subset V_1 \subset \cdots \subset V_{m_1} = V$. It is easy to check that $c_i = \dim V_i / V_{i-1}$. It is important to observe that the flag of subspaces $V_0 \subset V_1 \subset \cdots \subset V_{m_1}$ is naturally determined by A—no choice is involved. This is in contrast to the cyclic decomposition $V = W_1 \oplus \cdots \oplus W_r$ which can be chosen in many different ways.

Let $S \in S_A(k)$ and let $(p_1, \ldots, p_l)$ be its cyclic structure ($p_1 \geqslant \cdots \geqslant p_l \geqslant 1$, and $p_1 + \cdots + p_l = k$). Let $(q_1, \ldots, q_d)$ be the conjugate partition of $(p_1, \ldots, p_l)$. Let A_S be the restriction of A to S, and let $S_i \equiv \ker A_S^i$, $i = 0, 1, \ldots, p_1$. Then $S_i = S \cap V_i$, so $q_i = \dim S_i / S_{i-1} = \dim S \cap V_i / S \cap V_{i-1}$. Define a mapping $\phi \colon S \cap V_i / S \cap V_{i-1} \to V_i / V_{i-1}$ by $\phi(s + S \cap V_{i-1}) \equiv s + V_{i-1}$. Since $S \cap V_{i-1} \subseteq V_{i-1}$, ϕ is well defined. If $\phi(s + S \cap V_{i-1}) = 0 + V_{i-1}$, then $s \in V_{i-1}$. Since $s \in S \cap V_i$, this means that $s \in S \cap V_{i-1}$, so ϕ is injective. Since ϕ is a monomorphism, it follows that $q_i = \dim S \cap V_i / S \cap V_{i-1} \leqslant \dim V_i / V_{i-1} = c_i$, $\forall i$. It follows from the Ferrers diagram that $l \leqslant r$ and $p_j \leqslant m_j$, $j = 1, \ldots, l$.

Let k be fixed, and let $(p_1, \ldots, p_l)$ be a partition of k (with $p_1 \geqslant \cdots \geqslant p_l \geqslant 1$). We say that $(p_1, \ldots, p_l)$ is a *partition of k compatible with the block structure of A* if $l \leqslant r$ and $p_j \leqslant m_j$, $j = 1, \ldots, l$. Let $S_A(k; p_1, \ldots, p_l)$ be the subset of $S_A(k)$ consisting of those elements of $S_A(k)$ which have cyclic structure $(p_1, \ldots, p_l)$. The argument in the preceding paragraph shows that $S_A(k; p_1, \ldots, p_l)$ is empty unless $(p_1, \ldots, p_l)$ is compatible with the block structure of A. The converse is also true. If $(p_1, \ldots, p_l)$ is

compatible with the block structure of A, then

$$S \equiv \mathrm{Sp}\{e_{m_1-p_1+1}, \ldots, e_{m_1}, e_{m_1+m_2-p_2+1}, \ldots, e_{m_1+m_2}, \ldots,$$

$$e_{m_1+\cdots+m_l-p_l+1}, \ldots, e_{m_1+\cdots+m_l}\}$$

is an element of $S_A(k; p_1, \ldots, p_l)$. We state this as a proposition.

PROPOSITION 2. $S_A(k; p_1, \ldots, p_l)$ *is nonempty iff* $(p_1, \ldots, p_l)$ *is compatible with the block structure of* A.

There are two special cases where the structure of $S_A(k)$ is readily apparent and well known.

THEOREM 4. (a) *If* A *is semisimple* (*diagonalizable*), *then* $S_A(k) = G^k(\mathcal{F}^n)$ (*the Grassmann manifold of k-dimensional subspaces of* $\mathcal{F}^n$).
  (b) *If* A *is cyclic, then* $S_A(k)$ *consists of exactly one point.*

PROOF. (a) Since A is both nilpotent and semisimple, $A = 0$. Hence, every k-dimensional subspace of $\mathcal{F}^n$ is A-invariant, so $S_A(k) = G^k(\mathcal{F}^n)$. (Note that the block structure of A is $(m_1, \ldots, m_n) = (1, 1, \ldots, 1)$, and the only partition of k which is compatible with this block structure is $(p_1, \ldots, p_k) = (1, 1, \ldots, 1)$.)
  (b) Suppose that A is cyclic. Then the block structure of A is $(m_1) = (n)$. The only partition of k which is compatible with this block structure is $(p_1) = (k)$. Thus, every element of $S_A(k)$ is cyclic. Let $S \in S_A(k)$. Since S is cyclic, $\dim S \cap V_i = i$, $i = 1, \ldots, k$. In particular, $S = S \cap V_k$. Since A is cyclic, $\dim V_i = i$, $i = 1, \ldots, n$. Since $\dim V_k = k = \dim S$, it follows that $S = V_k$. Hence, $S_A(k) = \{V_k\}$. $\square$

PROPOSITION 3. *Let* $k \le r$. *The elements of* $S_A(k; 1, 1, \ldots, 1)$ *are the k-dimensional subspaces of* V_1. $S_A(k; 1, 1, \ldots, 1)$ *is a regular submanifold of* $G^k(\mathcal{F}^n)$ *which is isomorphic to* $G^k(\mathcal{F}^r)$.

PROOF. Let S be a k-dimensional subspace of V_1. Since $V_1 \equiv \ker A$, $A(S) = 0$, so S is A-invariant. Let $(p_1, \ldots, p_l)$ be the cyclic structure of S, and let $(q_1, \ldots, q_d)$ be its conjugate. Since $S \cap V_0 = 0$ and $S \cap V_i = S$ for $i > 0$, the formula $q_i = \dim S \cap V_i / S \cap V_{i-1}$ implies that $q_1 = k$, $q_i = 0$ for $i > 1$ (so $d = 1$). This means that $l = k$ and $p_1 = p_2 = \cdots = p_k = 1$. Thus, $S \in S_A(k; 1, 1, \ldots, 1)$. Conversely, suppose that $S \in S_A(k; 1, 1, \ldots, 1)$. Then $d = 1$ and $q_1 = k$, so $k = \dim S \cap V_1 / S \cap V_0 = \dim S \cap V_1$. Thus, $S \subseteq V_1$. This shows that $S_A(k; 1, 1, \ldots, 1)$ consists of all k-dimensional subspaces of V_1. Since $\dim V_1 = r$, $S_A(k; 1, 1, \ldots, 1) \approx G^k(\mathcal{F}^r)$. $\square$

PROPOSITION 4. *If* A *is cyclic, then* S_A *consists of exactly* $(n + 1)$ *points. Otherwise,* S_A *contains a connected component which is a projective space of positive dimension.*

PROOF. The first assertion follows immediately from Theorem 4. To prove the second assertion, assume that A is not cyclic. $S_A(1) = S_A(1; 1) \approx G^1(\mathcal{F}^r)$ by Proposition 3. Since A is not cyclic, $r \ge 2$, so $S_A(1)$ is a projective space of dimension $(r - 1) \ge 1$. $\square$

Theorem 4 describes the structure of S_A in the cases where A is semisimple or cyclic. These are the extreme cases. In the semisimple case, A has n 1×1 blocks,

whereas in the cyclic case A has one $n \times n$ block. A class of nilpotent operators which includes both the semisimple and the cyclic cases is the set of operators with block structure of the form $(m_1, \ldots, m_r) = (m_1, 1, \ldots, 1)$. (Note that $n = m_1 + r - 1$.) These are the operators which have at most one block of size greater than 1×1. Note that if $r = n$ and $m_1 = 1$, then A is semisimple, while if $r = 1$ and $m_1 = n$, then A is cyclic. We can describe explicitly the geometric structure of S_A for this class of operators.

LEMMA 1. *Let W_1 and W_2 be subspaces of V of dimensions n_1 and n_2 such that $W_1 \subseteq W_2 \subseteq V$. Suppose that $n_1 \leqslant k \leqslant n_2$. Let $M \equiv \{S \in G^k(V): W_1 \subseteq S \subseteq W_2\}$. Then M is a regular submanifold of $G^k(V)$ which is isomorphic to $G^{k-n_1}(\mathcal{F}^{n_2-n_1})$.*

PROOF. Choose an $(n_2 - n_1)$-dimensional subspace U such that $W_1 \oplus U = W_2$. Define a mapping $\phi: G^{k-n_1}(U) \to G^k(V)$ by $\phi(X) \equiv W_1 \oplus X$. Then ϕ is clearly an embedding with image M. Since $G^{k-n_1}(U) \approx G^{k-n_1}(\mathcal{F}^{n_2-n_1})$, the lemma is proved.

THEOREM 5. *Let A have block structure $(m_1, \ldots, m_r) \equiv (m_1, 1, \ldots, 1)$, and let $(p_1, \ldots, p_l) \equiv (p_1, 1, \ldots, 1)$ be a partition of k which is compatible with the block structure of A. Then*

  (i) *if $p_1 = 1$ or $l = r$, $S_A(k; p_1, 1, \ldots, 1) \approx G^l(\mathcal{F}^r)$,*

  (ii) *if $p_1 > 1$ and $l < r$, then $S_A(k; p_1, 1, \ldots, 1) \approx G^l(\mathcal{F}^r) - G^l(\mathcal{F}^{r-1})$, $\overline{S_A(k; p_1, 1, \ldots, 1)} \approx G^l(\mathcal{F}^r)$, and*

$$\overline{S_A(k; p_1, 1, \ldots, 1)} - S_A(k; p_1, 1, \ldots, 1) \subseteq \underbrace{S_A(k; p_1 - 1, 1, \ldots, 1)}_{(l+1) \text{ terms}}.$$

(*Overbar indicates closure.*)

PROOF. Let $S \in S_A(k; p_1, 1, \ldots, 1)$. Then $A \mid S$ is nilpotent of index p_1, so $S \subseteq V_{p_1}$ but $S \nsubseteq V_{p_1-1}$. Let $(c_1, \ldots, c_{m_1})$ and $(q_1, \ldots, q_{p_1})$ be the conjugates of $(m_1, \ldots, m_r)$ and $(p_1, \ldots, p_l)$ respectively. Then $(c_1, \ldots, c_{m_1}) = (r, 1, \ldots, 1)$ and $(q_1, \ldots, q_{p_1}) = (l, 1, \ldots, 1)$. It is easy to check that

$$\dim A(V_{p_1}) = \dim V_{p_1} - c_1 = \sum_{j=1}^{p_1} c_j - c_1 = \sum_{j=2}^{p_1} c_j = p_1 - 1.$$

Similarly, $\dim A(S) = \sum_{j=2}^{p_1} q_j = p_1 - 1$. Since $A(S) \subseteq A(V_{p_1})$ and $\dim A(S) = p_1 - 1 = \dim A(V_{p_1})$, it follows that $A(S) = A(V_{p_1})$. Since S is A-invariant, this implies that $A(V_{p_1}) \subseteq S$. Thus, if $S \in S_A(k; p_1, 1, \ldots, 1)$, then S is a k-dimensional subspace of V with $A(V_{p_1}) \subseteq S \subseteq V_{p_1}$ and $S \nsubseteq V_{p_1-1}$. Conversely, suppose S satisfies these conditions. Then $A(S) \subseteq A(V_{p_1}) \subseteq S$, so S is A-invariant. Since $S \subseteq V_{p_1}$ but $S \nsubseteq V_{p_1-1}$, $A \mid S$ is nilpotent of index p_1. Since the cyclic structure of S must be compatible with the block structure of A, S must have cyclic structure $(p_1, 1, \ldots, 1)$. Thus, $S \in S_A(k; p_1, 1, \ldots, 1)$. Hence, $S_A(k; p_1, 1, \ldots, 1) = \{S \in G^k(V): A(V_{p_1}) \subseteq S \subseteq V_{p_1}$ and $S \nsubseteq V_{p_1-1}\}$.

Let $M \equiv \{S \in G^k(V): A(V_{p_1}) \subseteq S \subseteq V_{p_1}\}$ and let $N \equiv \{S \in G^k(V): A(V_{p_1}) \subseteq S \subseteq V_{p_1-1}\}$. Since $\dim V_{p_1} - \dim A(V_{p_1}) = c_1 = r$ and $k - \dim A(V_{p_1}) = k - p_1 + 1 = l$, Lemma 1 implies that M is a submanifold of $G^k(V)$ isomorphic to $G^l(\mathcal{F}^r)$. If

$p_1 = 1$, then $V_{p_1-1} = 0$ and N is empty. (We are assuming that $k > 0$. $S_A(0)$ consists of one point for any operator A.) If $p_1 > 1$, then $\dim V_{p_1-1} = \sum_{j=1}^{p_1-1} c_j = r + p_1 - 2$. Since $k = p_1 + l - 1$, $\dim V_{p_1-1} = r + k - l - 1$. Thus, $k \leq \dim V_{p_1-1}$ iff $l < r$. If $l = r$, then N is empty. Otherwise, Lemma 1 implies that N is a submanifold of $G^k(V)$ isomorphic to $G^l(\mathscr{F}^{r-1})$. ($\dim V_{p_1-1} - \dim A(V_{p_1}) = \dim V_{p_1} - c_{p_1} - \dim A(V_{p_1}) = \dim V_{p_1} - \dim A(V_{p_1}) - c_{p_1} = r - 1$.) In fact, it is clear from its definition that N is a regular submanifold of M. Since $S_A(k; p_1, 1, \ldots, 1) = M - N$, we have showed that

$$S_A(k; p_1, 1, \ldots, 1) \approx \begin{cases} G^l(\mathscr{F}^r) & \text{if } p_1 = 1 \text{ or } l = r, \\ G^l(\mathscr{F}^r) - G^l(\mathscr{F}^{r-1}) & \text{otherwise.} \end{cases}$$

It remains only to prove the assertions regarding the limit points of $S_A(k; p_1, 1, \ldots, 1)$ in the case where $p_1 > 1$ and $l < r$. Since $S_A(k; p_1, 1, \ldots, 1) = M - N$ with M closed in $G^k(V)$ and N a submanifold of M of codimension $l > 0$, it follows that

$$\overline{S_A(k; p_1, 1, \ldots, 1)} = M \approx G^l(\mathscr{F}^r)$$

and

$$\overline{S_A(k; p_1, 1, \ldots, 1)} - S_A(k; p_1, 1, \ldots, 1) = N.$$

Suppose that $S \in N$. Then $A(V_{p_1}) \subseteq S \subseteq V_{p_1-1}$. Then $A(S) \subseteq A(V_{p_1-1}) \subseteq A(V_{p_1}) \subseteq S$, so S is A-invariant. The condition $A(V_{p_1}) \subseteq S \subseteq V_{p_1-1}$ implies that $A \mid S$ is nilpotent of index $p_1 - 1$. Hence, $S \in S_A(k; p_1 - 1, 1, \ldots, 1)$. Thus, $N \subseteq S_A(k; p_1 - 1, 1, \ldots, 1)$. This completes the proof. $\square$

REMARK 3. If the block structrure of A is $(m_1, \ldots, m_r) \equiv (m_1, 1, \ldots, 1)$, and $(p_1, \ldots, p_l)$ is a partition of k compatible with the block structure of A, then $p_2 = p_3 = \cdots = p_l = 1$. Thus, Theorem 5 describes $S_A(k; p_1, \ldots, p_l)$ for every choice of $(p_1, \ldots, p_l)$ which is compatible with the block structure of A.

REMARK 4. Theorem 5 shows that if A has block structure of the form $(m_1, \ldots, m_r) = (m_1, 1, \ldots, 1)$, then the geometry of $S_A(k; p_1, 1, \ldots, 1)$ depends only on l, the number of terms in the partition of k. Specifically, if $p_1 > 1$ and $p_1 + j \leq m_1$, then $S_A(k; p_1, 1, \ldots, 1)$ has the same structure as $S_A(k + j; p_1 + j, 1, \ldots, 1)$. This is not true for an operator with arbitrary block structure $(m_1, \ldots, m_r)$. It is easy to show this using the results in the next section. For example, we will prove that $S_A(k; k)$ has dimension $\sum_{i=1}^{k} c_i - k$. Thus,

$$\dim S_A(k + j; k + j) = \sum_{i=1}^{k+j} c_i - (k + j).$$

This shows that $\dim S_A(k; k) = \dim S_A(k + j; k + j)$ iff $c_{k+1} = \cdots = c_{k+j} = 1$. This is true when the block structure of A is $(m_1, 1, \ldots, 1)$, but it is not true in general.

Let $P(k)$ be the set of all partitions of k. If $p \equiv (p_1, \ldots, p_l)$ and $p' \equiv (p'_1, \ldots, p'_{l'})$ are two elements of $P(k)$, we say that $p \lesssim p'$ iff $l \geq l'$ and $\sum_{i=1}^{j} p_i \leq \sum_{i=1}^{j} p'_i$, $j = 1, \ldots, l'$. This defines a partial order on $P(k)$ which we refer to as the *natural*

ordering. If q and q' are the conjugates of p and p' respectively, it can be shown that $p \lesssim p'$ iff $q \gtrsim q'$ [2]. Thus, conjugation is order-reversing.

Let $P_A(k)$ be the subset of $P(k)$ consisting of all partitions of k which are compatible with the block structure of A. By using a Ferrers diagram, it is easy to see that $P_A(k)$ contains a largest element, p^*, and a smallest element, p_*. To determine p^*, draw the Ferrers diagram for $(m_1, \ldots, m_r)$ and mark k of the boxes, proceeding row by row starting at the upper left-hand corner of the diagram. The marked boxes give the partition p^*. To determine p_*, mark k of the boxes starting in the upper left-hand corner, but proceed column by column.

EXAMPLE 2. Let $(m_1, m_2, m_3) = (5, 3, 1)$. Let $k = 6$. Then p^* and p_* are determined by the following diagrams:

$$p^* = (5, 1) \qquad\qquad p_* = (3, 2, 1)$$
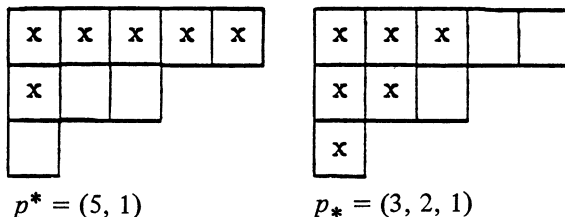
We will use the natural ordering as a tool to prove that $S_A(k)$ is connected. Our approach is to show that if $p_* \equiv (p_1, \ldots, p_l)$, then $S_A(k; p_1, \ldots, p_l)$ is Grassmannian and hence connected. Then we show that if $S \in S_A(k)$, there is a continuous curve in $S_A(k)$ which starts at S and which intersects the connected subset $S_A(k; p_1, \ldots, p_l)$.

PROPOSITION 5. *Let* $p_* \equiv (p_1, \ldots, p_l)$ *be the smallest element of* $P_A(k)$, *and let* $(q_1, \ldots, q_d)$ *be its conjugate. Then* $S_A(k; p_1, \ldots, p_l)$ *is a submanifold of* $G^k(V)$ *isomorphic to* $G^{q_d}(\mathscr{F}^{c_d})$. *(Note that* $d = p_1$.)

PROOF. Let $S \in S_A(k; p_1, \ldots, p_l)$. From the method of computing p_* using the Ferrers diagram for $(m_1, \ldots, m_r)$, it is clear that $q_j = c_j$ for $j \leqslant d - 1$. This implies that S contains V_j for $j \leqslant d - 1$. Since $q_{d+1} = 0$, $S \subseteq V_d$. Thus S is a k-dimensional subspace of V such that $V_{d-1} \subseteq S \subseteq V_d$. Conversely, suppose that S is such a subspace. Then $A(S) \subseteq A(V_d) \subseteq V_{d-1} \subseteq S$, so S is A-invariant. Let $S \in S_A(k; p'_1, \ldots, p'_{l'})$, and let $(q'_1, \ldots, q'_{d'})$ be the conjugate of $(p'_1, \ldots, p'_{l'})$. Since $V_{d-1} \subseteq S$, it follows that $q'_j = c_j$ for $j \leqslant d - 1$. Since $S \subseteq V_d$, it follows that $d' \leqslant d$. Thus $d' \leqslant d$ and $q'_j = q_j$ for $j \leqslant d - 1$. Since $q'_1 + \cdots + q'_{d'} = k = q_1 + \cdots + q_d$, this implies that $d' = d$ and $q'_d = q_d$ as well. So $S \in S_A(k; p_1, \ldots, p_l)$. Thus, $S_A(k; p_1, \ldots, p_l) = \{S \in G^k(V): V_{d-1} \subseteq S \subseteq V_d\}$. Since $\dim V_d - \dim V_{d-1} = c_d$ and $k - \dim V_{d-1} = q_d$, the conclusion follows immediately from Lemma 1. $\square$

COROLLARY. *If* $p_* = (p_1, \ldots, p_l)$, *then* $S_A(k; p_1, \ldots, p_l)$ *is path-connected.*

REMARK 5. In the next section we prove that $S_A(k; p_1, \ldots, p_l)$ is connected for any $(p_1, \ldots, p_l) \in P_A(k)$.

LEMMA 2. *Let* $p \equiv (p_1, \ldots, p_l)$ *be any element of* $P_A(k)$ *other than* p_*. *Let* $S_0 \in S_A(k; p_1, \ldots, p_l)$. *Then there exists a partition* $p' \equiv (p'_1, \ldots, p'_{l'}) \in P_A(k)$ *with* $p' < p$ *and a continuous curve* $\gamma: [0, 1] \to S_A(k)$ *such that* $\gamma(0) = S_0$, $\gamma(t) \in S_A(k; p_1, \ldots, p_l)$ *for* $t < 1$, *and* $\gamma(1) \in S_A(k; p'_1, \ldots, p'_{l'})$.

PROOF. Let $(q_1, \ldots, q_{p_1})$ be the conjugate partition of $(p_1, \ldots, p_l)$. Let $\nu \equiv \min\{i: q_i < c_i\}$. Using the Ferrers diagram for $(m_1, \ldots, m_r)$, it is easy to see that the assumption that $p \neq p_*$ implies that $\nu < p_1$. From the definition of ν, it follows that $V_{\nu-1} \subseteq S_0$, but $V_\nu \not\subseteq S_0$. Choose $w \in V_\nu$ such that $\omega \notin S_0$. Choose a basis for S_0 of the form $\{v_1, Av_1, \ldots, A^{p_1-1}v_1; v_2, Av_2, \ldots, A^{p_2-1}v_2; \ldots; v_l, Av_l, \ldots, A^{p_l-1}v_l\}$. Let

$$\gamma(t) \equiv \mathrm{Sp}\big\{(\cos \tfrac{1}{2}\pi t)v_1 + (\sin \tfrac{1}{2}\pi t)w, Av_1, \ldots, A^{p_1-1}v_1;$$

$$v_2, Av_2, \ldots, A^{p_2-1}v_2; \ldots; v_l, Av_l, \ldots, A^{p_l-1}v_l\big\}.$$

To show that $\gamma(t)$ is A-invariant, it clearly suffices to show that $Aw \in \gamma(t)$. Now, $Aw \in A(V_\nu) \subseteq V_{\nu-1} \subseteq S_0$, so $Aw = \sum_{i=1}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j-1} v_i$. Applying $A^{\nu-1}$ to both sides gives

$$0 = A^\nu w = \sum_{i=1}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j+\nu-2} v_i.$$

By linear independence, it follows that if $A^{j+\nu-2}v_i \neq 0$, then $\alpha_{ij} = 0$. Since $\nu < p_1$, $A^{\nu-1}v_1 \neq 0$, so $\alpha_{11} = 0$. But this shows that $Aw \in \gamma(t)$. Hence, $\gamma(t)$ is A-invariant. Since $w \notin S_0$, the indicated spanning set for $\gamma(t)$ is linearly independent. Hence $\dim \gamma(t) = k$, so $\gamma(t) \in S_A(k)$.

Suppose that $t < 1$. Let $x(t) \equiv (\cos \tfrac{1}{2}\pi t)v_1 + (\sin \tfrac{1}{2}\pi t)w$. Let $\beta(t) \equiv \mathrm{Sp}\{x(t), Ax(t), \ldots, A^{p_1-1}x(t); v_2, Av_2, \ldots, A^{p_2-1}v_2; \ldots; v_l, Av_l, \ldots, A^{p_l-1}v_l\}$. Since $\gamma(t)$ is A-invariant and contains $x(t)$, it also contains $A^j x(t)$. Hence, $\beta(t) \subseteq \gamma(t)$. To show that $\beta(t) = \gamma(t)$, it suffices to prove that $\dim \beta(t) = k$, which is the same as showing that the indicated spanning set for $\beta(t)$ is linearly independent. Suppose there exists an equation of the form

$$(*) \qquad \sum_{j=1}^{p_1} \alpha_{1j} A^{j-1} x(t) = -\sum_{i=2}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j-1} v_i.$$

Applying A^{p_1-1} to both sides of $(*)$ gives $\alpha_{11} A^{p_1-1} x(t) = -\sum_{i=2}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j+p_1-2} v_i$. Since $A^\nu w = 0$ and $p_1 - 1 \geq \nu$, this becomes

$$\alpha_{11} A^{p_1-1}(\cos \tfrac{1}{2}\pi t) v_1 = -\sum_{i=2}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j+p_1-2} v_i.$$

Since $t < 1$, $\cos \tfrac{1}{2}\pi t \neq 0$. Hence, if $\alpha_{11} \neq 0$, this equation is a nontrivial linear dependence among the vectors in the original basis for S_0, a contradiction. Thus, $\alpha_{11} = 0$. Now, apply A^{p_1-2} to both sides of $(*)$. Since $\alpha_{11} = 0$, this gives $\alpha_{12} A^{p_1-1}(\cos \tfrac{1}{2}\pi t) v_1 = -\sum_{i=2}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j+p_1-3} v_i$, which implies that $\alpha_{12} = 0$. Continuing in this way, we obtain $\alpha_{11} = \alpha_{12} = \cdots = \alpha_{1p_1} = 0$. But then $(*)$ is a linear dependence among basis vectors for S_0, so each coefficient, α_{ij}, must be zero. Thus, the indicated spanning set is actually a basis for $\beta(t)$. This proves that $\beta(t) = \gamma(t)$ (for $t < 1$), showing that the cyclic structure of $\gamma(t)$ is $(p_1, \ldots, p_l)$—i.e., $\gamma(t) \in S_A(k; p_1, \ldots, p_l)$ for $t < 1$. (It is definitely not true that $\beta(1) = \gamma(1)$. In fact, $\beta(1)$ need not be k-dimensional.)

Now

$$\gamma(1) = \mathrm{Sp}\big\{w, Av_1, \ldots, A^{p_1-1}v_1; v_2, Av_2, \ldots, A^{p_2-1}v_2; \ldots; v_l, Av_l, \ldots, A^{p_l-1}v_l\big\}.$$

Let $(p_1', \ldots, p_{l'}')$ be the cyclic structure of $\gamma(1)$, and let $(q_1', \ldots, q_{p_1'}')$ be its conjugate. Let $M_0 \equiv \mathrm{Sp}\{Av_1, \ldots, A^{p_1-1}v_1; v_2, Av_2, \ldots, A^{p_2-1}v_2; \ldots; v_l, Av_l, \ldots, A^{p_l-1}v_l\}$. Since $S_0 \cap V_i \subseteq M_0$ for $i < p_1$ and $M_0 \subseteq \gamma(1)$, it follows that $S_0 \cap V_i \subseteq \gamma(1) \cap V_i$ for $i < p_1$. Since $S_0 \cap V_i = V_i$ for $i < \nu$, this implies that $\gamma(1) \cap V_i = V_i$ for $i < \nu$, which means that $q_i' = c_i = q_i$ for $i < \nu$.

Now suppose that $\nu \leqslant s < p_1$. $\gamma(1) \cap V_s$ contains $S_0 \cap V_s$ and w, so $(S_0 \cap V_s) \oplus \mathrm{Sp}\{w\} \subseteq \gamma(1) \cap V_s$. Suppose $z \in \gamma(1) \cap V_s$. Then we can express z as

$$z \equiv \lambda w + \sum_{j=2}^{p_1} \alpha_{1j} A^{j-1}v_1 + \sum_{i=2}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j-1}v_i.$$

Since $z \in V_s$,

$$0 = A^s z = \sum_{j=2}^{p_1} \alpha_{1j} A^{j+s-1}v_1 + \sum_{i=2}^{l} \sum_{j=1}^{p_i} \alpha_{ij} A^{j+s-1}v_i,$$

where we have used the fact that $A^s w = 0$. By linear independence, it follows that $\alpha_{ij} = 0$ if $A^{j+s-1}v_i \neq 0$. Thus, if the expression for z includes a nonzero multiple of $A^{j-1}v_i$, then $A^s(A^{j-1}v_i) = 0$ which implies that $A^{j-1}v_i \in S_0 \cap V_s$. Thus, $z \in (S_0 \cap V_s) \oplus \mathrm{Sp}\{w\}$. Hence, $\gamma(1) \cap V_s = (S_0 \cap V_s) \oplus \mathrm{Sp}\{w\}$ if $\nu \leqslant s < p_1$. Finally, $\gamma(1) \cap V_{p_1} = \gamma(1)$. Thus

$$\gamma(1) \cap V_i = \begin{cases} V_i = S_0 \cap V_i & \text{for } i < \nu, \\ (S_0 \cap V_i) \oplus \mathrm{Sp}\{w\} & \text{for } \nu \leqslant i < p_1, \\ \gamma(1) & \text{for } i = p_1. \end{cases}$$

This implies that $q_i' = q_i$ for $i < \nu$, $q_\nu' = q_\nu + 1$, $q_i' = q_i$ for $\nu < i < p_1$, and $q_{p_1}' = q_{p_1} - 1$. In particular, $q' > q$. Since conjugation is order-reversing, this implies that $p' < p$. $\square$

COROLLARY. *Let* $p \equiv (p_1, \ldots, p_l) \in P_A(k)$. *Then* $S_A(k; p_1, \ldots, p_l)$ *is closed in* $G^k(V)$ *iff* $p = p_*$.

PROOF. If $p = p_*$, then $S_A(k; p_1, \ldots, p_l)$ is a Grassmannian submanifold of $G^k(V)$ (Proposition 5) and is therefore closed. If $p \neq p_*$, then Lemma 2 implies that $S_A(k; p_1, \ldots, p_l)$ is not closed. $\square$

THEOREM 6. $S_A(k)$ *is path-connected.*

PROOF. Let $p_* \equiv (p_1, \ldots, p_l)$. Let $S_0 \in S_A(k)$. Since $P_A(k)$ is a finite set, we can apply Lemma 2 successively to obtain a continuous curve $\gamma: [0, 1] \to S_A(k)$ with $\gamma(0) = S_0$ and $\gamma(1) \in S_A(k; p_1, \ldots, p_l)$. Since $S_A(k; p_1, \ldots, p_l)$ is itself path-connected (Corollary to Proposition 5), the same is true of $S_A(k)$. $\square$

5. Differentiable structure of $S_A(k; p_1, \ldots, p_l)$. In this section, we investigate the geometric structure of $S_A(k; p_1, \ldots, p_l)$. This exploration culminates in Theorem 7, which shows that $S_A(k; p_1, \ldots, p_l)$ is an analytic manifold which embeds in $G^k(\mathfrak{F}^n)$.

Let $F(n, k)$ be the set of $n \times k$ matrices of rank k with entries in $\mathfrak{F}$. Then $F(n, k)$ is open in $\mathfrak{F}^{nk}$. $Gl(k, \mathfrak{F})$ acts freely on $F(n, k)$ on the right by matrix multiplication, and the orbit space $F(n, k)/Gl(k, \mathfrak{F})$ is the Grassmann manifold $G^k(\mathfrak{F}^n)$ of

k-dimensional linear subspaces of $\mathcal{F}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a multi-index (i.e., $\alpha_1, \ldots, \alpha_k$ are integers such that $1 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqslant n$). For each α and each $F \in F(n, k)$, let $M_\alpha(F)$ be the $k \times k$ submatrix formed by rows $\alpha_1, \ldots, \alpha_k$ of F. Let $U_\alpha \equiv \{F \in F(n, k): \det M_\alpha(F) \neq 0\}$. Then $\{U_\alpha\}$ is a cover of $F(n, k)$ by open dense subsets. For each α, define $\eta_\alpha: F(n, k) \to \mathcal{F}^{(n-k)k}$ to be the mapping which deletes the $k \times k$ submatrix $M_\alpha(F)$ from F. Define $\phi_\alpha: U_\alpha \to \mathcal{F}^{(n-k)k}$ by $\phi_\alpha(F) \equiv \eta_\alpha(F \cdot M_\alpha(F)^{-1})$. Let $\pi: F(n, k) \to F(n, k)/Gl(k, \mathcal{F})$ be the natural projection. ϕ_α induces an injective mapping $\overline{\phi}_\alpha$ of $\overline{U}_\alpha \equiv \pi(U_\alpha)$ onto $\mathcal{F}^{(n-k)k}$ such that $\phi_\alpha = \overline{\phi}_\alpha \circ \pi$. $\{(\overline{U}_\alpha, \overline{\phi}_\alpha)\}$ is an atlas for $F(n, k)/Gl(k, \mathcal{F})$ which has regular functions for the change of coordinates. This atlas gives $G^k(\mathcal{F}^n)$ the structure of an $\mathcal{F}$-analytic manifold of dimension $(n - k)k$. The charts in this atlas are called the standard charts for $G^k(\mathcal{F}^n)$.

The results of the preceding section show that, in general, $S_A(k)$ is not a regular submanifold of $G^k(\mathcal{F}^n)$. In particular, there can exist points $S_1, S_2 \in S_A(k)$ which have neighborhoods U_1, U_2 which are homeomorphic to open sets in affine spaces of unequal dimensions. This raises the question of whether or not there are subsets of $S_A(k)$ which are submanifolds of $G^k(\mathcal{F}^n)$. In this section, we show that $S_A(k; p_1, \ldots, p_l)$ is a regular submanifold of $G^k(\mathcal{F}^n)$. Our approach is to construct $S_A(k; p_1, \ldots, p_l)$ as the orbit space of a particular submanifold of $F(n, k)$ under the free action of a closed subgroup of $Gl(k, \mathcal{F})$. This construction is analogous to, but more difficult than, the construction of $G^k(\mathcal{F}^n)$ as $F(n, k)/Gl(k, \mathcal{F})$. Finally, we show that the orbit space embeds in $G^k(\mathcal{F}^n)$ in a natural way.

Throughout this section, A is a fixed $n \times n$ nilpotent matrix in lower Jordan form with block sizes $(m_1, \ldots, m_r)$. Let $(p_1, \ldots, p_l)$ be a partition of k compatible with the block structure of A. Let $(c_1, \ldots, c_s)$ and $(q_1, \ldots, q_d)$ be the conjugate partitions of $(m_1, \ldots, m_r)$ and $(p_1, \ldots, p_l)$ respectively.

Let $S \in S_A(k; p_1, \ldots, p_l)$. Then S has an ordered basis of the form $\{v_1, Av_1, \ldots, A^{p_1-1}v_1, v_2, Av_2, \ldots, A^{p_2-1}v_2, \ldots, v_l, Av_l, \ldots, A^{p_l-1}v_l\}$, with $A^{p_i}v_i = 0$, $i = 1, \ldots, l$. Such an ordered basis will be called a *cyclic basis* for S. Let B be the $n \times k$ matrix whose columns are the vectors in this ordered basis. Partition the rows of B according to the partition $(m_1, \ldots, m_r)$ of n, and partition the columns according to the partition $(p_1, \ldots, p_l)$ of k. Then B consists of rl blocks, and the ijth block, B_{ij}, is $m_i \times p_j$. It is easy to verify directly that B_{ij} has the following structure: (i) B_{ij} is constant along diagonals; (ii) if the diagonals of B_{ij} are numbered starting with the *lower* left-hand corner, and if a_t is the constant value of the entries on the tth diagonal ($t = 1, \ldots, m_i + p_j - 1$), then $a_t = 0$ for $t > \min(m_i, p_j)$.

A matrix with properties (i) and (ii) will be called *regular lower triangular* (RLT). The following are examples of RLT matrices.

$$\begin{pmatrix} a_3 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ a_2 & 0 \\ a_1 & a_2 \end{pmatrix}, \quad \begin{pmatrix} a_2 & 0 & 0 \\ a_1 & a_2 & 0 \end{pmatrix}.$$

A partitioned matrix whose blocks are RLT matrices will be called *block regular lower triangular* (BRLT). Thus, if B is a matrix whose columns form a cyclic basis for

a subspace $S \in S_A(k; p_1, \ldots, p_l)$, then B is a full rank $n \times k$ matrix which is BRLT when partitioned according to $(m_1, \ldots, m_r)$ and $(p_1, \ldots, p_l)$. Conversely, if B is such a matrix, it is clear that the columns of B form a cyclic basis for some $S \in S_A(k; p_1, \ldots, p_l)$.

Let $\mathscr{B}(k; p_1, \ldots, p_l)$ be the set of $n \times k$ rank k matrices with entries in $\mathscr{F}$ which are BRLT when partitioned according to $(m_1, \ldots, m_r)$ and $(p_1, \ldots, p_l)$. The preceding observations are summarized in the following proposition.

PROPOSITION 6. *Let* $(p_1, \ldots, p_l)$ *be a partition of k compatible with the block structure of A. Then for each* $S \in S_A(k; p_1, \ldots, p_l)$ *there exists* $B \in \mathscr{B}(k; p_1, \ldots, p_l)$ *such that* $\mathrm{Sp}\, B = M$. *Conversely, if* $B \in \mathscr{B}(k; p_1, \ldots, p_l)$, *then* $\mathrm{Sp}\, B \in S_A(k; p_1, \ldots, p_l)$.

REMARK 6. The notation $\mathscr{B}(k; p_1, \ldots, p_l)$ suppresses the row partition $(m_1, \ldots, m_r)$ corresponding to the block structure of A. Since A is fixed throughout the discussion, this should cause no confusion.

REMARK 7. If $(p_1, \ldots, p_l)$ is a partition of k which is not compatible with $(m_1, \ldots, m_r)$, then no $n \times k$ matrix which is BRLT with respect to $(m_1, \ldots, m_r)$ and $(p_1, \ldots, p_l)$ can have rank k. If B were such a matrix, then $\mathrm{Sp}\, B$ would be an A-invariant subspace with cyclic structure $(p_1, \ldots, p_l)$ not compatible with the block structure of A, which is impossible.

REMARK 8. Let $(p_1, \ldots, p_l)$ be a fixed partition of k which is compatible with $(m_1, \ldots, m_r)$. Let R be the $k \times k$ nilpotent matrix in lower Jordan form with block structure $(p_1, \ldots, p_l)$. Then it is easy to verify that the elements of $\mathscr{B}(k; p_1, \ldots, p_l)$ are the rank k solutions of the linear matrix equation $AX = XR$, where X is an $n \times k$ unknown matrix [3, pp. 215–220].

EXAMPLE 3. Let $(m_1, m_2, m_3) = (5, 3, 1)$. Then the elements of $\mathscr{B}(5; 3, 2)$ are the full rank matrices of the form

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 0 \\
x_4 & x_3 & 0 & y_4 & 0 \\
x_5 & x_4 & x_3 & y_5 & y_4 \\
\hline
x_6 & 0 & 0 & 0 & 0 \\
x_7 & x_6 & 0 & y_7 & 0 \\
x_8 & x_7 & x_6 & y_8 & y_7 \\
\hline
x_9 & 0 & 0 & y_9 & 0
\end{bmatrix}.
$$

PROPOSITION 7. $\mathscr{B}(k; p_1, \ldots, p_l)$ *is a regular submanifold of $F(n, k)$ of dimension* $\sum_{i=1}^{d} c_i q_i$.

PROOF. It is obvious from its definition that $\mathscr{B}(k; p_1, \ldots, p_l)$ is a regular submanifold of $F(n, k)$. Let $B \in \mathscr{B}(k; p_1, \ldots, p_l)$. B contains rl blocks, and block B_{ij} is $m_i \times p_j$. Since B_{ij} is RLT, it contains $\min(m_i, p_j)$ parameters which are arbitrary

except for the condition that B be full rank. Thus

$$\dim \mathfrak{B}(k; p_1,\ldots,p_l) = \sum_{i=1}^{r} \sum_{j=1}^{l} \min(m_i, p_j).$$

To express this sum more simply, note that there are c_i blocks whose minimum dimension is at least i. Thus, there are $(c_i q_i - c_{i+1}q_{i+1})$ blocks whose minimum dimension is exactly i. So $\dim \mathfrak{B}(k; p_1,\ldots,p_l) = \sum_{i=1}^{d}(c_i q_i - c_{i+1}q_{i+1})i$. Using the fact that $q_{d+1} = 0$, this sum simplifies to $\sum_{i=1}^{d} c_i q_i$. $\square$

Let $(p_1,\ldots,p_l)$ be a fixed partition of k. Define R as in Remark 8. Let $G(k; p_1,\ldots,p_l)$ be the subgroup of $Gl(k,\mathfrak{F})$ consisting of those matrices which commute with R. Applying Remark 8 to the special case where the block structure of A is $(p_1,\ldots,p_l)$ (and hence $A = R$) shows that $G(k; p_1,\ldots,p_l)$ consists of the nonsingular $k \times k$ matrices with entries in $\mathfrak{F}$ which are BRLT when both the rows and the columns are partitioned according to $(p_1,\ldots,p_l)$.

PROPOSITION 8. $G(k; p_1,\ldots,p_l)$ is a closed subgroup of $Gl(k,\mathfrak{F})$ of dimension $\sum_{i=1}^{d} q_i^2$.

PROOF. It is clear from the structure of the matrices in $G(k; p_1,\ldots,p_l)$ that they form a closed subgroup of $Gl(k,\mathfrak{F})$. The dimension formula follows from Proposition 7 applied to the special case where $n = k$ and the block structure of A is $(p_1,\ldots,p_l)$.

PROPOSITION 9. $G(k; p_1,\ldots,p_l)$ acts freely on $\mathfrak{B}(k; p_1,\ldots,p_l)$ on the right by matrix multiplication. Let $\mathfrak{B}(k; p_1,\ldots,p_l)/G(k; p_1,\ldots,p_l)$ be the quotient space, and let $\pi: \mathfrak{B}(k; p_1,\ldots,p_l) \to \mathfrak{B}(k; p_1,\ldots,p_l)/G(k; p_1,\ldots,p_l)$ be the natural projection. Then $\pi(B_1) = \pi(B_2)$ iff $\mathrm{Sp}\, B_1 = \mathrm{Sp}\, B_2$.

PROOF. Let $B \in \mathfrak{B}(k; p_1,\ldots,p_l)$ and $Z \in G(k; p_1,\ldots,p_l)$. Then BZ has rank k and $A(BZ) = (AB)Z = (BR)Z = B(RZ) = B(ZR) = (BZ)R$ since $AB = BR$ and Z commutes with R. So by Remark 8, $BZ \in \mathfrak{B}(k; p_1,\ldots,p_l)$. Thus, $G(k; p_1,\ldots,p_l)$ acts on $\mathfrak{B}(k; p_1,\ldots,p_l)$. If $BZ = B$, then $Z = I$ since B has full rank, so the action is free. If $\pi(B_1) = \pi(B_2)$, then there exists $Z \in G(k; p_1,\ldots,p_l)$ with $B_1 Z = B_2$, so $\mathrm{Sp}\, B_1 = \mathrm{Sp}\, B_2$. Conversely, suppose $\mathrm{Sp}\, B_1 = \mathrm{Sp}\, B_2$. Then there exists $Y \in Gl(k,\mathfrak{F})$ such that $B_1 Y = B_2$. Since $AB_1 = B_1 R$ and $AB_2 = B_2 R$, $B_1(RY) = (B_1 R)Y = (AB_1)Y = A(B_1 Y) = AB_2 = B_2 R = (B_1 Y)R = B_1(YR)$. Since B_1 is rank k, this implies that $RY = YR$. Hence, $Y \in G(k; p_1,\ldots,p_l)$, so $\pi(B_1) = \pi(B_2)$. $\square$

REMARK 9. Proposition 9 shows that $B_1, B_2 \in \mathfrak{B}(k; p_1,\ldots,p_l)$ span the same A-invariant subspace iff there exists $Z \in G(k; p_1,\ldots,p_l)$ such that $B_1 Z = B_2$. Thus, the action of $G(k; p_1,\ldots,p_l)$ on $\mathfrak{B}(k; p_1,\ldots,p_l)$ corresponds to change of cyclic basis in k-dimensional A-invariant subspaces of cyclic structure $(p_1,\ldots,p_l)$. Each element of $\mathfrak{B}(k; p_1,\ldots,p_l)/G(k; p_1,\ldots,p_l)$ is an equivalence class consisting of all the cyclic bases for a single subspace in $S_A(k; p_1,\ldots,p_l)$.

LEMMA 3. Let $\sim$ be an equivalence relation on a topological space X such that $\pi: X \to X/\sim$ is an open mapping. Let $E \equiv \{(x_1, x_2) \in X \times X: x_1 \sim x_2\}$. Then $X/\sim$ is Hausdorff iff E is closed in $X \times X$.

PROOF. (See [1, p. 61].)

PROPOSITION 10. $\mathfrak{B}(k;\ p_1,\ldots,p_l)/G(k;\ p_1,\ldots,p_l)$ is a Hausdorff topological space (in the quotient topology) with a countable basis.

PROOF. To streamline notation, let $\mathfrak{B} \equiv \mathfrak{B}(k;\ p_1,\ldots,p_l)$ and let $G \equiv G(k;\ p_1,\ldots,p_l)$. Since the action of G on $\mathfrak{B}$ is continuous, π is an open mapping. Since $\mathfrak{B}$ has a countable basis, the same is true for $\mathfrak{B}/G$. Let $D \equiv \{(B_1, B_2) \in \mathfrak{B} \times \mathfrak{B}: \exists Z \in G \text{ such that } B_1 Z = B_2\}$. Let $E \equiv \{(X_1, X_2) \in F(n, k) \times F(n, k): \exists Y \in Gl(k, \mathfrak{F}) \text{ such that } X_1 Y = X_2\}$. By the preceding lemma, $\mathfrak{B}/G$ is Hausdorff iff D is closed in $\mathfrak{B} \times \mathfrak{B}$. Since $F(n, k)/Gl(k, \mathfrak{F})$ is Hausdorff, E is closed in $F(n, k) \times F(n, k)$. Since G is a subgroup of $Gl(k, \mathfrak{F})$, $D \subseteq E \cap (\mathfrak{B} \times \mathfrak{B})$. On the other hand, if $(X_1, X_2) \in E \cap \mathfrak{B} \times \mathfrak{B}$, then $X_1, X_2 \in \mathfrak{B}$ and $\text{Sp } X_1 = \text{Sp } X_2$. From Proposition 9 it follows that $\pi(B_1) = \pi(B_2)$, so $(X_1, X_2) \in D$. Thus, $D = E \cap (\mathfrak{B} \times \mathfrak{B})$. Since $\mathfrak{B} \times \mathfrak{B}$ has the relative topology from $F(n, k) \times F(n, k)$, D is closed in $\mathfrak{B} \times \mathfrak{B}$. Hence, $\mathfrak{B}/G$ is Hausdorff. $\square$

In order to develop coordinate charts for $\mathfrak{B}(k;\ p_1,\ldots,p_l)/G(k;\ p_1,\ldots,p_l)$, it is necessary to consider the action of $G(k;\ p_1,\ldots,p_l)$ in detail. Let $B \in \mathfrak{B}(k;\ p_1,\ldots,p_l)$. Let $B \equiv [b_1^1,\ldots,b_{p_1}^1;\ b_1^2,\ldots,b_{p_2}^2;\ldots;\ b_1^l,\ldots,b_{p_l}^l]$ be the partition of B by columns. Each of the l sets of columns is called a *block of columns*. Consider the following three types of transformations.

(i) Multiply each of the p_j columns in the jth block of the columns by a nonzero $\lambda \in \mathfrak{F}$.

$$\left[b_1^1,\ldots,b_{p_1}^1;\ldots;b_1^j,\ldots,b_{p_j}^j;\ldots;b_1^l,\ldots,b_{p_l}^l\right]$$
$$\to \left[b_1^1,\ldots,b_{p_1}^1;\ldots;\lambda b_1^j,\ldots,\lambda b_{p_j}^j;\ldots;b_1^l,\ldots,b_{p_l}^l\right].$$

(ii) Take the *last* ν columns in the jth block of columns, multiply each of these by $\lambda \in \mathfrak{F}$, and add them consecutively to the *first* ν columns in the ith block of columns.

$$\left[b_1^1,\ldots,b_{p_1}^1;\ldots;b_1^i,\ldots,b_{p_i}^i;\ldots;b_1^j,\ldots,b_{p_j}^j;\ldots;b_1^l,\ldots,b_{p_l}^l\right]$$
$$\to \Big[b_1^1,\ldots,b_{p_1}^1;\ldots;b_1^i + \lambda b_{p_j-\nu+1}^j,\ b_2^i + \lambda b_{p_j-\nu+2}^j,\ldots,b_\nu^i + \lambda b_{p_j}^j,$$
$$b_{\nu+1}^i,\ldots,b_{p_i}^i;\ldots;b_1^j,\ldots,b_{p_j}^j;\ldots;b_1^l,\ldots,b_{p_l}^l\Big].$$

(Obviously, $\nu \leqslant p_i$. Also, it is allowable to choose $i = j$. However, if this choice is made, we require that $\nu < p_j$. The case where $i = j$ and $\nu = p_j$ is included in (i).)

(iii) If $p_i = p_j$, interchange the ith and jth blocks of columns.

$$\left[b_1^1,\ldots,b_{p_1}^1;\ldots;b_1^i,\ldots,b_{p_i}^i;\ldots;b_1^j,\ldots,b_{p_j}^j;\ldots;b_1^l,\ldots,b_{p_l}^l\right]$$
$$\to \left[b_1^1,\ldots,b_{p_1}^1;\ldots;b_1^j,\ldots,b_{p_j}^j;\ldots;b_1^i,\ldots,b_{p_i}^i;\ldots;b_1^l,\ldots,b_{p_l}^l\right].$$

These transformations can be accomplished by multiplication by the elements of $G(k;\ p_1,\ldots,p_l)$ described below. (Z_{uv} is the uvth block of $Z \in G(k;\ p_1,\ldots,p_l)$.)

(i) $Z_{uv} = 0$ for $u \neq v$; $Z_{uu} = I$ for $u \neq j$; $Z_{jj} = \lambda I$.

(ii) If $i \neq j$, then $Z_{uu} = i$ for all u; Z_{ji} is zero except for the νth diagonal (counting from the lower left-hand corner) which is equal to λ; every other block is zero. If $i = j$, then $Z_{uu} = I$ for $u \neq j$; Z_{jj} is one on the p_jth diagonal, λ on the νth diagonal, and zero elsewhere; $Z_{uv} = 0$ for $u \neq v$.

(iii) $Z_{ij} = Z_{ji} = I$; $Z_{uu} = I$ for $u \neq i, j$; all other blocks are zero.

A transformation of type (i), (ii), or (iii) is called an *elementary cyclic column operation* (ECCO). The corresponding element of $G(k; p_1, \ldots, p_l)$ is called an *elementary matrix* (EM). Later it wil be proved that every matrix in $G(k; p_1, \ldots, p_l)$ is a product of elementary matrices. (It is true that the EM's of types (i) and (ii) generate $G(k; p_1, \ldots, p_l)$. However, it is convenient to include type (iii) matrices in the class of EM's even though every result could be obtained without using them.)

Let $B \in \mathfrak{B}(k; p_1, \ldots, p_l)$. Let (i, j) be fixed. Suppose that the p_jth diagonal (counting from the lower left-hand corner) of block B_{ij} is nonzero, say λ. Multiply each column in the jth block of columns (in B) by $1/\lambda$. This is an ECCO of type (i), and the p_jth diagonal of the ijth block in the resulting matrix is equal to one. For convenience use "B" to denote the new matrix. Now suppose that the $(p_j - 1)$th diagonal of (the new) B_{ij} is nonzero, say λ. Multiply by $-\lambda$ the *last* $(p_j - 1)$ columns in the jth block of columns (in V) and add them consecutively to the *first* $(p_j - 1)$ columns in the jth block of columns. This is an ECCO of type (ii), and the $(p_j - 1)$th diagonal of the new B_{ij} is zero. The p_jth diagonal of B_{ij} remains equal to one. Now suppose that the $(p_j - 2)$th diagonal of (the new) B_{ij} is nonzero, say λ. Multiply by $-\lambda$ the last $(p_j - 2)$ columns in the jth block of columns and add them consecutively to the first $(p_j - 2)$ columns in the jth block of columns. The new B_{ij} has diagonal p_j equal to one and diagonals $(p_j - 1)$ and $(p_j - 2)$ equal to zero. By continuing in this way, we obtain a new matrix B such that the p_jth diagonal of B_{ij} is one and the other diagonals of B_{ij} are zero.

Now consider block $B_{ij'}$ (in the transformed matrix). Suppose that the p_jth diagonal of $B_{ij'}$ is nonzero, say λ. Multiply by $-\lambda$ the p_j columns in the jth block of columns and add them consecutively to the first p_j columns in the j'th block of columns. The p_jth diagonal in $B_{ij'}$ is now zero. Now suppose that the $(p_j - 1)$th diagonal of $B_{ij'}$ is nonzero, say λ. Multiply by $-\lambda$ the last $(p_j - 1)$ columns in the jth block of columns and add them consecutively to the first $(p_j - 1)$ columns in the j'th block of columns. The new $B_{ij'}$ has both diagonals p_j and $(p_j - 1)$ equal to zero. Continuing in this way, we eliminate the first p_j diagonals of $B_{ij'}$. Next we eliminate the first p_j diagonals of $B_{ij''}$ etc. Eventually we obtain a transformed matrix B such that diagonals $1, \ldots, p_j$ of $B_{i1}, B_{i2}, \ldots, B_{il}$ are all zero except that diagonal p_j of B_{ij} is one. This result is summarized in the following lemma.

LEMMA 4. *Let $B \in \mathfrak{B}(k; p_1, \ldots, p_l)$. Let (i, j) be fixed. Suppose that the p_jth diagonal of B_{ij} is nonzero. Then by applying a sequence of ECCO's, B can be transformed into a matrix $\hat{B}$ such that the first p_j diagonals of $\hat{B}_{i1}, \ldots, \hat{B}_{il}$ are all zero except that the p_jth diagonal of $\hat{B}_{ij}$ is one.*

Let $\gamma = (\gamma_1, \ldots, \gamma_l)$ be a multi-index of length l such that $\gamma_1, \ldots, \gamma_l$ are distinct integers between 1 and r. They need not be in increasing order. We say that γ is

compatible iff $m_{\gamma_j} \geqslant p_j$, $j = 1, \ldots, l$. Since $(p_1, \ldots, p_l)$ is compatible with the block structure of A, $m_j \geqslant p_j$, $j = 1, \ldots, l$, so $\gamma \equiv (1, 2, \ldots, l)$ is always a compatible multi-index. Let $\Gamma(k; p_1, \ldots, p_l)$ be the set of compatible multi-indices corresponding to the partition $(p_1, \ldots, p_l)$ of k. (As usual, the partition $(m_1, \ldots, m_r)$ of n is suppressed.) For each $\gamma \in \Gamma(k; p_1, \ldots, p_l)$ and each $B \in \mathcal{B}(k; p_1, \ldots, p_l)$, let $M_\gamma(B)$ be the $k \times k$ submatrix formed by taking the last p_j rows from the γ_jth block of rows, $j = 1, \ldots, l$.

LEMMA 5. *Let* $B \in \mathcal{B}(k; p_1, \ldots, p_l)$. *Then there exists* $\gamma \in \Gamma(k; p_1, \ldots, p_l)$ *such that* $\det M_\gamma(B) \neq 0$.

PROOF. Consider the blocks $B_{11}, \ldots, B_{r1}$ in the first column of blocks in B. B_{i1} is $m_i \times p_1$. Since each block is RLT, it is zero above the p_1th diagonal. If the p_1th diagonal of B_{i1} is also zero, then the last column in B_{i1} is identically zero. If this is true for $B_{11}, \ldots, B_{r1}$, then the p_1th column in B is identically zero, which is impossible since B has full rank. Thus one of these blocks, say $B_{\gamma_1,1}$, has a nonzero p_1th diagonal. By Lemma 4, a sequence of ECCO's can be applied to B to make the first p_1 diagonals in $B_{\gamma_1,1}, \ldots, B_{\gamma_1,l}$ equal to zero except for the p_1th diagonal of $B_{\gamma_1,1}$ which is equal to one. Since $B_{\gamma_1,j}$ is automatically zero above the p_jth diagonal (since it is RLT), and since $p_1 \geqslant p_j$, this means that $B_{\gamma_1,1}, \ldots, B_{\gamma_1,l}$ are zero except for the p_1th diagonal of $B_{\gamma_1,1}$ which is one.

Now consider the second column of blocks, $B_{12}, \ldots, B_{r2}$, in the new matrix. Since the $(p_1 + p_2)$th column of B is not identically zero, one of these blocks, say $B_{\gamma_2,2}$, has a nonzero p_2th diagonal. Since $B_{\gamma_1,2}$ is zero, $\gamma_2 \neq \gamma_1$. By Lemma 4, a sequence of ECCO's can be applied to B to make the first p_2 diagonals in $B_{\gamma_2,1}, \ldots, B_{\gamma_2,l}$ equal to zero except for the p_2th diagonal of $B_{\gamma_2,2}$ which is equal to one. Since $B_{\gamma_2,j}$ is automatically zero above the p_jth diagonal, and since $p_2 \geqslant p_j$ if $j \geqslant 2$, this means that $B_{\gamma_2,2}, \ldots, B_{\gamma_2,l}$ are zero except for the p_2th diagonal of $B_{\gamma_2,2}$ which is one. Also, it is not hard to see that since $B_{\gamma_1,2}$ is zero, the ECCO's used to put $B_{\gamma_2,1}, \ldots, B_{\gamma_2,l}$ into the described form do not change the entries in $B_{\gamma_1,1}, \ldots, B_{\gamma_1,l}$. Thus, the form of $B_{\gamma_1,1}, \ldots, B_{\gamma_1,l}$ is not disturbed.

Continuing in this way, we obtain a compatible multi-index $\gamma \equiv (\gamma_1, \ldots, \gamma_l)$ and a transformed matrix, call it $\hat{B}$, such that the first p_j diagonals of $\hat{B}_{\gamma_j,1}, \ldots, \hat{B}_{\gamma_j,l}$ are zero except for the p_jth diagonal of $\hat{B}_{\gamma_j,j}$ which is one. $\hat{B}_{\gamma_j,j}, \ldots, \hat{B}_{\gamma_j,l}$ are zero except for the p_jth diagonal of $\hat{B}_{\gamma_j,j}$ which is one.

Form the $k \times k$ submatrix $M_\gamma(\hat{B})$. Both the rows and the columns of $M_\gamma(\hat{B})$ are naturally partitioned according to $(p_1, \ldots, p_l)$. Let $\{M_{ij}\}$ be the blocks in $M_\gamma(\hat{B})$. M_{ij} is $p_i \times p_j$ and consists of the last p_i rows of $\hat{B}_{\gamma_i,j}$. Thus M_{jj} is an identity matrix, and M_{ij} is a zero matrix if $i < j$. Thus, $\det M_\gamma(\hat{B}) = 1$. Since $\hat{B}$ is obtained from B by ECCO's there exists $Z \in G(k; p_1, \ldots, p_l)$ such that $\hat{B} = BZ$. Then $M_\gamma(\hat{B}) = M_\gamma(B)Z$, so $\det M_\gamma(B) \neq 0$. $\square$

EXAMPLE 4. Let $(m_1, m_2, m_3) = (5, 3, 1)$. Let $k = 5$ and $(p_1, p_2) = (3, 2)$. The elements of $\mathcal{B}(5; 3, 2)$ were described in Example 3. $\Gamma(5; 3, 2) = \{(1, 2), (2, 1)\}$. Let

$B \in \mathcal{B}(5; 3, 2)$. By Lemma 5, at least one of the submatrices

$$
\begin{pmatrix}
x_3 & 0 & 0 & 0 & 0 \\
x_4 & x_3 & 0 & y_4 & 0 \\
x_5 & x_4 & x_3 & y_5 & y_4 \\
\hline
x_7 & x_6 & 0 & y_7 & 0 \\
x_8 & x_7 & x_6 & y_8 & y_7
\end{pmatrix}
,\quad
\begin{pmatrix}
x_6 & 0 & 0 & 0 & 0 \\
x_7 & x_6 & 0 & y_7 & 0 \\
x_8 & x_7 & x_6 & y_8 & y_7 \\
\hline
x_4 & x_3 & 0 & y_4 & 0 \\
x_5 & x_4 & x_3 & y_5 & y_4
\end{pmatrix}
,
$$

corresponding to the multi-indices $(1, 2)$ and $(2, 1)$ respectively must be nonsingular. Note that if $x_6 \neq 0$, the first submatrix is not BRLT. Similarly, if $x_3 \neq 0$, the second submatrix is not BRLT.

Let M be an $r \times s$ matrix which is constant on diagonals and is zero above the sth diagonal. Then we say that M is *regular weakly lower triangular* (RWLT). If $s \leq r$, then RWLT is equivalent to RLT. However, if $r < s$, then RWLT is a less stringent condition than RLT. For example,

$$
\begin{pmatrix}
b & c & 0 \\
a & b & c
\end{pmatrix}
$$

is RWLT but is not RLT unless $c = 0$. A partitioned matrix whose blocks are RWLT is called *block regular weakly lower triangle* (BRWLT).

Let $H(k; p_1, \ldots, p_l)$ be the set of all nonsingular $k \times k$ matrices which are BRWLT when both the rows and columns are partitioned according to $(p_1, \ldots, p_l)$. It is easy to check that $H(k; p_1, \ldots, p_l)$ is invariant under each of the three types of ECCO's. It is also clear that $G(k; p_1, \ldots, p_l)$ is a regular submanifold of $H(k; p_1, \ldots, p_l)$.

Let $M \in H(k; p_1, \ldots, p_l)$, and let M_{ij} be the ijth block of M ($i, j = 1, \ldots, l$). We say that M is in *standard form* iff the first p_i diagonals of $M_{i1}, \ldots, M_{il}$ are all zero except for the p_ith diagonal of M_{ii} which is one. Note that since M_{ij} is $p_i \times p_j$ and is RWLT, the diagonals above the p_jth are automatically zero. Thus, if M is in standard form, then M_{ii} is a $p_i \times p_i$ identity matrix and $M_{ij} = 0$ for $i < j$.

EXAMPLE 5. The elements of $H(5; 3, 2)$ are the nonsingular matrices of the form

$$
\begin{pmatrix}
x_{11} & 0 & 0 & 0 & 0 \\
x_{21} & x_{11} & 0 & x_{24} & 0 \\
x_{31} & x_{21} & x_{11} & x_{34} & x_{24} \\
\hline
x_{41} & x_{42} & 0 & x_{44} & 0 \\
x_{51} & x_{41} & x_{42} & x_{54} & x_{44}
\end{pmatrix}
.
$$

The elements of $H(5; 3, 2)$ which are in standard form are of the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\hline
0 & x_{42} & 0 & 1 & 0 \\
0 & 0 & x_{42} & 0 & 1
\end{pmatrix}
.
$$

LEMMA 6. *Let* $M \in H(k; p_1, \ldots, p_l)$. *Then there exists a matrix* $\hat{M} \in H(k; p_1, \ldots, p_l)$ *which is standard form and a sequence of ECCO's which transform* M *into* $\hat{M}$.

PROOF. Suppose first that $p_1 > p_2$. Consider the blocks in the first row of blocks in M, namely $M_{11}, \ldots, M_{1l}$. Since M_{1j} is RWLT, it is zero above the p_jth diagonal. Since $p_1 > p_2$, this means that $M_{12}, \ldots, M_{1l}$ are zero above the $(p_1 - 1)$th diagonal. This implies that the top row of M_{1j} is zero for $j = 2, \ldots, l$. Since M is nonsingular, the top row of M_{11} cannot be zero, for otherwise the first row of M would be identically zero. This means that the p_1th diagonal of M_{11} is nonzero. Noting that the proof of Lemma 4 applies to $H(k; p_1, \ldots, p_l)$ just as well as it applies to $\mathfrak{B}(k; p_1, \ldots, p_l)$, we conclude that there exists a sequence of ECCO's which can be applied to M to make the first p_1 diagonals of $M_{11}, \ldots, M_{1l}$ zero except for the p_1th diagonal of M_{11} which is one. Since M_{1j} is zero above the p_jth diagonal, and $p_1 \geqslant p_j$, it follows that M_{11} is a $p_1 \times p_1$ identity matrix and $M_{12}, \ldots, M_{1l}$ are all zero.

Now suppose that $p_1 = p_2 = \cdots = p_\nu > p_{\nu+1}$. Since $p_1 > p_{\nu+1}$, it follows that $M_{1,\nu+1}, \ldots, M_{1l}$ are zero above the $(p_1 - 1)$th diagonal. Thus, the top row of each of $M_{1,\nu+1}, \ldots, M_{1l}$ is zero. Since the top row of M is not identically zero, as least one of $M_{11}, \ldots, M_{1\nu}$ has a nonzero top row and hence a nonzero p_1th diagonal. By performing an ECCO of type (iii) if necessary, we can assume that the p_1th diagonal of M_{11} is nonzero. Then we can proceed as in the case where $p_1 > p_2$. Thus in either case we obtain the indicated form for the first row of blocks.

Next, consider the second row of blocks in the (new) matrix M. Since M_{11} is an identity matrix and $M_{12}, \ldots, M_{1l}$ are zero matrices, it is clear that for det M to be nonzero, the top rows of $M_{22}, \ldots, M_{2l}$ cannot all be zero. If $p_2 > p_3$, this implies that the p_2th diagonal of M_{22} is nonzero. If $p_2 = p_3 = \cdots = p_\nu > p_{\nu+1}$, we perform an ECCO of type (iii) (if necessary) to make the p_2th diagonal of M_{22} nonzero. Then by Lemma 4, there exists a sequence of ECCO's which makes the first p_2 diagonals of $M_{21}, \ldots, M_{2l}$ zero except for the p_2th diagonal of M_{22} which is one. Since $M_{12}, \ldots, M_{1l}$ are zero matrices, these ECCO's do not change any entries in the first row of blocks. Since M_{2j} is a $p_2 \times p_j$ RWLT matrix, it is automatically zero above the p_jth diagonal. Since $p_2 \geqslant p_j$ if $2 \leqslant j$, this implies that B_{22} is a $p_2 \times p_2$ identity matrix and $B_{23}, \ldots, B_{2l}$ are all zero.

Continuing in this way, we obtain a sequence of ECCO's which transform M into a matrix $\hat{M}$ which is in standard form. $\square$

COROLLARY. *Every element of* $G(k; p_1, \ldots, p_l)$ *is a product of EM's.*

PROOF. Let $Z \in G(k; p_1, \ldots, p_l)$. By Lemma 6, there exists a matrix $\hat{Z} \in H(k; p_1, \ldots, p_l)$ which is in standard form and a sequence $Z_1, \ldots, Z_m$ of EM's such that $\hat{Z} = ZZ_1Z_2 \cdots Z_m$. Since $Z, Z_1, \ldots, Z_m \in G(k; p_1, \ldots, p_l)$ and $G(k; p_1, \ldots, p_l)$ is a group, it follows that $\hat{Z} \in G(k; p_1, \ldots, p_l)$. But the elements of $G(k; p_1, \ldots, p_l)$ are BRLT—not just BRWLT. From the definition of standard form, it is clear that the only matrix in $H(k; p_1, \ldots, p_l)$ which is BRLT and in standard form is I. So

$I = \hat{Z} = ZZ_1Z_2 \cdots Z_m$. Hence, $Z^{-1} = Z_1Z_2 \cdots Z_m$. Thus Z^{-1} is a product of EM's. Since Z is arbitrary, this shows that every element of $G(k; p_1,\ldots,p_l)$ is a product of EM's. □

LEMMA 7. *Let* $M, N \in H(k; p_1,\ldots,p_l)$ *be in standard form, and suppose there exists* $Z \in G(k; p_1,\ldots,p_l)$ *such that* $N = MZ$. *Then* $Z = I$ *and* $N = M$.

PROOF. For convenience, let $LT(X)$ denote the lower triangular part of a matrix X. This proof is by induction on l.

Let M_{ij}, N_{ij}, Z_{ij} be the ijth blocks of M, N, Z respectively. Suppose that $l = 1$. Then $N = N_{11} = I$ and $M = M_{11} = I$, so $Z = I$ and the result is trivial. Suppose the result holds for $l = 1,\ldots,\nu - 1$. Let $l = \nu$. Since $N = MZ$, $N_{1j} = \sum_{r=1}^{l} M_{1r}Z_{rj}$. Since $M_{11} = I$ and $M_{12},\ldots,M_{1l}$ are zero, this becomes $N_{1j} = Z_{1j}$. Since $N_{11} = I$ and $N_{12},\ldots,N_{1l}$ are zero, this means that $Z_{11} = I$ and $Z_{12},\ldots,Z_{1l}$ are zero.

Now, $N_{21} = \sum_{r=1}^{l} M_{2r}Z_{r1} = M_{21}Z_{11} + M_{22}Z_{21} = M_{21} + Z_{21}$. (Here we have used the facts that $M_{22} = I$, $M_{23},\ldots,M_{2l}$ are zero, and $Z_{11} = I$.) Taking the lower triangular part of this equation gives $LT(N_{21}) = LT(M_{21}) + LT(Z_{21})$, which gives $0 = 0 + LT(Z_{21})$. Hence $LT(Z_{21}) = 0$. Since Z_{21} is RLT, this implies that $Z_{21} = 0$. Next, consider the equation

$$N_{31} = \sum_{r=1}^{l} M_{3r}Z_{r1} = M_{31}Z_{11} + M_{32}Z_{21} + M_{33}Z_{31} = M_{31} + Z_{31}.$$

Hence, $0 = LT(N_{31}) = LT(M_{31}) + LT(Z_{31}) = LT(Z_{31})$, so $Z_{31} = 0$. Continuing in this way gives $Z_{41},\ldots,Z_{l1} = 0$. Thus, the equation $MZ = N$ can be written as

$$\left(\begin{array}{c|ccc} M_{11} & M_{12} \cdots M_{1l} \\ \hline M_{21} & \\ \vdots & \tilde{M} \\ M_{l1} & \end{array}\right) \left(\begin{array}{c|ccc} I & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \tilde{Z} \\ 0 & \end{array}\right) = \left(\begin{array}{c|ccc} N_{11} & N_{12} \cdots N_{1l} \\ \hline N_{21} & \\ \vdots & \tilde{N} \\ N_{l1} & \end{array}\right).$$

This implies that $\tilde{M}\tilde{Z} = \tilde{N}$. Clearly, $\tilde{M}, \tilde{N} \in H(k - p_1; p_2,\ldots,p_l)$ and are in standard form, and $\tilde{Z} \in G(k - p_1; p_2,\ldots,p_l)$. Thus, the induction hypothesis may be applied to conclude that $\tilde{Z} = I$. Hence, $Z = I$ and the proof is complete. □

By Lemmas 6 and 7, it follows that for each $M \in H(k; p_1,\ldots,p_l)$ there exists a unique $\hat{M} \in H(k; p_1,\ldots,p_l)$ which is in standard form and a unique $Z \in G(k; p_1,\ldots,p_l)$ such that $\hat{M} = MZ$. From the proof of Lemma 6, it is clear that the entries in $\hat{M}$ are regular functions of the entries in M. Since $Z = M^{-1}\hat{M}$, the entries in Z are regular functions of the entries in M. Define $\sigma: H(k; p_1,\ldots,p_l) \to G(k; p_1,\ldots,p_l)$ with $\sigma(M) = M^{-1}\hat{M}$. Note that if $M \in G(k; p_1,\ldots,p_l)$, then $\hat{M} = I$, so $\sigma(M) = M^{-1}$. Thus, the restriction of σ to $G(k; p_1,\ldots,p_l)$ is the inversion mapping.

Let $\pi: \mathcal{B}(k; p_1,\ldots,p_l) \to \mathcal{B}(k; p_1,\ldots,p_l)/G(k; p_1,\ldots,p_l)$ and $\pi': F(n, k) \to F(n, k)/Gl(k, \mathcal{F})$ be the natural projections, and let $i: \mathcal{B}(k; p_1,\ldots,p_l) \to F(n, k)$ be the inclusion map. Let $B_1, B_2 \in \mathcal{B}(k; p_1,\ldots,p_l)$. By Proposition 9, $\pi(B_1) = \pi(B_2)$ iff $Sp\, B_1 = Sp\, B_2$. But $Sp\, B_1 = Sp\, B_2$ iff $\pi' \circ i(B_1) = \pi' \circ i(B_2)$. Thus, there exists

an injection $\mu\colon \mathcal{B}/G \to F(n, k)/Gl(k, \mathcal{F})$ such that $\mu \circ \pi = \pi' \circ i$—i.e., such that the diagram

$$
\begin{array}{ccc}
\mathcal{B} & \overset{i}{\to} & F(n, k) \\
\pi \downarrow & & \downarrow \pi' \\
\mathcal{B}/G & \underset{\mu}{\to} & F(n, k)/Gl(k, \mathcal{F})
\end{array}
$$

is commutative. By Proposition 6, the image of μ is precisely $S_A(k;\, p_1, \ldots, p_l)$. This leads to the principal result of this section.

THEOREM 7. $\mathcal{B}(k;\, p_1, \ldots, p_l)/G(k;\, p_1, \ldots, p_l)$ is an $\mathcal{F}$-analytic manifold of dimension $\Sigma_{i=1}^d q_i(c_i - q_i)$. The induced mapping $\mu\colon \mathcal{B}/G \to F(n, k)/Gl(k, \mathcal{F})$ is an embedding with image $S_A(k;\, p_1, \ldots, p_l)$.

PROOF. For each $\gamma \in \Gamma(k;\, p_1, \ldots, p_l)$, let $V_\gamma \equiv \{B \in \mathcal{B}(k;\, p_1, \ldots, p_l)\colon \det M_\gamma(B) \neq 0\}$. Let $U_\gamma \equiv \{F \in F(n, k)\colon \det M_\gamma(F) \neq 0\}$. Since U_γ is open in $F(n, k)$ and $V_\gamma = U_\gamma \cap \mathcal{B}$, V_γ is open in the relative topology on $\mathcal{B}$. By Lemma 5, $\{V_\gamma\}_{\gamma \in \Gamma}$ is a cover of $\mathcal{B}$. Let $\overline{V}_\gamma \equiv \pi(V_\gamma)$. Since π is an open mapping, $\{\overline{V}_\gamma\}_{\gamma \in \Gamma}$ is an open cover of $\mathcal{B}/G$.

If $B \in V_\gamma$, B has $\Sigma_{i=1}^d c_i q_i$ diagonals which are not automatically zero (see Proposition 7). $B \cdot \sigma(M_\gamma(B))$ has $\Sigma_{i=1}^d q_i^2$ of these diagonals fixed at either one or zero and therefore contains $\Sigma_{i=1}^d q_i(c_i - q_i)$ nontrival diagonals. Let

$$
N \equiv \sum_{i=1}^d q_i(c_i - q_i).
$$

For each $\gamma \in \Gamma$, define a mapping $\eta_\gamma\colon \mathcal{B} \to \mathcal{F}^N$ which maps $B \in \mathcal{B}$ to the N-tuple consisting of its entries on the N diagonals just described (in some preassigned order). (η_γ is just a projection mapping which maps B to a vector composed of N of its entries.) Define a mapping $\psi_\gamma\colon V_\gamma \to \mathcal{F}^N$ by $\psi_\gamma(B) \equiv \eta_\gamma(B \cdot \sigma(M_\gamma(B)))$. Suppose that $B_1, B_2 \in V_\gamma$ with $\pi(B_1) = \pi(B_2)$. Then

$$
\pi\big(B_1 \cdot \sigma\big(M_\gamma(B_1)\big)\big) = \pi\big(B_2 \cdot \sigma\big(M_\gamma(B_2)\big)\big),
$$

so there exists $Z \in G$ such that $B_1 \cdot \sigma(M_\gamma(B_1)) \cdot Z = B_2 \cdot \sigma(M_\gamma(B_2))$. Thus, $M_\gamma(B_1 \cdot \sigma(M_\gamma(B_1))) \cdot Z = M_\gamma(B_2 \cdot \sigma(M_\gamma(B_2)))$. Since $M_\gamma(B_1 \cdot \sigma(M_\gamma(B_1)))$ and $M_\gamma(B_2 \cdot \sigma(M_\gamma(B_2)))$ are both in standard form, Lemma 7 implies that $Z = I$. Hence, $B_1 \cdot \sigma(M_\gamma(B_1)) = B_2 \cdot \sigma(M_\gamma(B_2))$. This shows that there exists $\overline{\psi}_\gamma\colon \overline{V}_\gamma \to \mathcal{F}^N$ such that $\overline{\psi}_\gamma \circ \pi = \psi_\gamma$.

Now suppose that $B_1 \cdot \sigma(M_\gamma(B_1)) = B_2 \cdot \sigma(M_\gamma(B_2))$. It follows immediately that $\pi(B_1) = \pi(B_2)$. Hence, the mapping of $\overline{V}_\gamma$ into V_γ given by $[B] \to B \cdot \sigma(M_\gamma(B))$ is one-to-one. From its definition it is clear that η_γ is one-to-one when restricted to the image of this mapping. Thus $\overline{\psi}_\gamma$ is injective. It is easy to see that $\overline{\psi}_\gamma$ maps $\overline{V}_\gamma$ homeomorphically onto $\mathcal{F}^N$. It is clear that if $\gamma, \gamma' \in \Gamma$ and $\overline{V}_\gamma \cap \overline{V}_{\gamma'} \neq \varnothing$, then each component of the mapping $\overline{\psi}_{\gamma'} \circ \overline{\psi}_\gamma^{-1}\colon \overline{\psi}_\gamma(\overline{V}_\gamma \cap \overline{V}_{\gamma'}) \to \overline{\psi}_{\gamma'}(\overline{V}_\gamma \cap \overline{V}_{\gamma'})$ is a regular function. Thus, $\{(\overline{V}_\gamma, \overline{\psi}_\gamma)\}_{\gamma \in \Gamma}$ is an atlas for $\mathcal{B}/G$ which gives it an $\mathcal{F}$-analytic structure. So $\mathcal{B}/G$ is an $\mathcal{F}$-analytic manifold of dimension $N = \Sigma_{i=1}^d q_i(c_i - q_i)$.

We have already showed that μ is a one-to-one mapping of $\mathfrak{B}/G$ onto $S_A(k; p_1,\ldots,p_l)$. It is trivially true that $\pi'(i(\mathfrak{B}) \cap U_\gamma) \subseteq \pi'(i(\mathfrak{B})) \cap \pi'(U_\gamma)$. Let $S \in \pi'(i(\mathfrak{B})) \cap \pi'(U_\gamma)$. Then there exist $B \in \mathfrak{B}$ and $F \in U_\gamma$ such that $S = \pi'(i(B)) = \pi'(F)$. But this implies that there exists $Y \in Gl(k, \mathfrak{F})$ such that $BY = F$. Since $\det M_\gamma(F) \neq 0$, it follows that $\det M_\gamma(B) \neq 0$, so $B \in U_\gamma$. Hence, $S \in \pi'(i(\mathfrak{B}) \cap U_\gamma)$. So $\pi'(i(\mathfrak{B}) \cap U_\gamma) = \pi'(i(\mathfrak{B})) \cap \pi'(U_\gamma)$. Then,

$$\mu(\overline{V}_\gamma) = \mu(\pi(V_\gamma)) = \pi'(i(V_\gamma)) = \pi'(i(\mathfrak{B}) \cap U_\gamma)$$
$$= \pi'(i(\mathfrak{B})) \cap \pi'(U_\gamma)$$
$$= \mu(\pi(\mathfrak{B})) \cap \overline{U}_\gamma = S_A(k; p_1,\ldots,p_l) \cap \overline{U}_\gamma.$$

The completion of this proof is postponed until after an example.

EXAMPLE 6. Let $(m_1, m_2, m_3) = (5, 3, 1)$. Let $k = 5$ and $(p_1, p_2) = (3, 2)$. Then from Example 4, $\Gamma(5; 3, 2)$ consists of two multi-indices, $\gamma \equiv (1, 2)$ and $\gamma' \equiv (2, 1)$. Let $B \in \mathfrak{B}(5; 3, 2)$. The form of B is described in Example 3 which shows that B contains 12 nontrivial diagonals. Suppose that $B \in V_\gamma$. Then the 5×5 submatrix $M_\gamma(B)$ is nonsingular. Multiplying B by $\sigma(M_\gamma(B))$ puts this submatrix in standard form. (This standard form is shown in Example 5.) Thus, $B \cdot \sigma(M_\gamma(B))$ has the form

$$B \cdot \sigma(M_\gamma(B)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \tilde{x}_6 & 0 & 0 & 0 & 0 \\ 0 & \tilde{x}_6 & 0 & 1 & 0 \\ 0 & 0 & \tilde{x}_6 & 0 & 1 \\ \hline \tilde{x}_9 & 0 & 0 & \tilde{y}_9 & 0 \end{bmatrix}.$$

This matrix contains only three nontrivial diagonals. $\tilde{x}_6, \tilde{x}_9, \tilde{y}_9$ are regular functions of the twelve variables $(x_3, x_4, x_5, x_6, x_7, x_8, x_9, y_4, y_5, y_7, y_8, y_9)$ in B. Then $\eta_\gamma(B \cdot \sigma(M_\gamma(B))) = (\tilde{x}_6, \tilde{x}_9, \tilde{y}_9)$. Thus, with B as in Example 3, $\psi_\gamma(B) = (\tilde{x}_6, \tilde{x}_9, \tilde{y}_9)$. The induced map $\overline{\psi}_\gamma$ is the mapping

$$\mathrm{Sp} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \tilde{x}_6 & 0 & 0 & 0 & 0 \\ 0 & \tilde{x}_6 & 0 & 1 & 0 \\ 0 & 0 & \tilde{x}_6 & 0 & 1 \\ \hline \tilde{x}_9 & 0 & 0 & \tilde{y}_9 & 0 \end{bmatrix} \overset{\overline{\psi}_\gamma}{\to} (\tilde{x}_6, \tilde{x}_9, \tilde{y}_9).$$

Now, the corresponding chart U_γ for $F(9,5)$ consists of all 9×5 matrices F such that $M_\gamma(F)$ is nonsingular. The usual coordinate system on the chart $\overline{U}_\gamma$ for $F(9,5)/Gl(5, \mathscr{F})$ is given by the mapping

$$
\text{Sp}
\begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\hline
f_{61} & f_{62} & f_{63} & f_{64} & f_{65} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\hline
f_{91} & f_{92} & f_{93} & f_{94} & f_{95}
\end{bmatrix}
\xrightarrow{\overline{\phi}_\gamma}
\begin{pmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\
f_{61} & f_{62} & f_{63} & f_{64} & f_{65} \\
f_{91} & f_{92} & f_{93} & f_{94} & f_{95}
\end{pmatrix}
$$

This is the coordinate system for $\overline{U}_\gamma$ described at the beginning of this section. However, it is trivial to check that the coordinate system $\overline{\phi}'_\gamma$ for $\overline{U}_\gamma$ defined by

$$
\text{Sp}
\begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\hline
f_{61} & f_{62} & f_{63} & f_{64} & f_{65} \\
0 & f_{61} & 0 & 1 & 0 \\
0 & 0 & f_{61} & 0 & 1 \\
\hline
f_{91} & f_{92} & f_{93} & f_{94} & f_{95}
\end{bmatrix}
\xrightarrow{\overline{\phi}'_\gamma}
\begin{pmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\
f_{61} & f_{62} & f_{63} & f_{64} & f_{65} \\
f_{91} & f_{92} & f_{93} & f_{94} & f_{95}
\end{pmatrix}
$$

is related to $\overline{\phi}_\gamma$ by a regular change of coordinates. Since

$$
\overline{\phi}'_\gamma \circ \mu \circ \overline{\psi}_\gamma^{-1}(\tilde{x}_6, \tilde{x}_9, \tilde{y}_9) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\tilde{x}_6 & 0 & 0 & 0 & 0 \\
\tilde{x}_9 & 0 & 0 & \tilde{y}_9 & 0
\end{pmatrix},
$$

μ is an immersion on $\overline{U}_\gamma$. Similarly, one shows that μ is an immersion on $\overline{U}_{\gamma'}$.

Now return to the proof of the theorem. It is easy to see that Example 6 illustrates the general situation. By modifying the usual coordinate system $\overline{\phi}_\gamma$ for $\overline{U}_\gamma$ by a regular change of coordinates, we obtain new coordinates $\overline{\phi}'_\gamma$ for $\overline{U}_\gamma$ such that $\overline{\phi}'_\gamma \circ \mu \circ \overline{\psi}_\gamma^{-1}$ is the canonical immersion of $\mathscr{F}^N$ into $\mathscr{F}^{(n-k)k}$. Thus, μ is a one-to-one immersion of $\mathscr{B}/G$ into $G^k(\mathscr{F}^n) \equiv F(n, k)/Gl(k, \mathscr{F})$ with image $S_A(k; p_1, \ldots, p_l)$. To prove that μ is an embedding, it remains only to show that μ^{-1} is continuous as a mapping from $S_A(k; p_1, \ldots, p_l)$ onto $\mathscr{B}/G$. Let $h_\gamma \equiv \overline{\phi}'_\gamma \circ \mu \circ \overline{\psi}_\gamma^{-1}$. h_γ maps $\mathscr{F}^N = \overline{\psi}_\gamma(\overline{V}_\gamma)$ injectively onto $\overline{\phi}'_\gamma \circ \mu(\overline{V}_\gamma) = \overline{\phi}'_\gamma(S_A(k; p_1, \ldots, p_l) \cap \overline{U}_\gamma)$, so h_γ^{-1} exists as a mapping from $\overline{\phi}'_\gamma(S_A(k; p_1, \ldots, p_l) \cap \overline{U}_\gamma)$ onto $\overline{\psi}_\gamma(\overline{V}_\gamma)$. Since h_γ is the canonical immersion of $\mathscr{F}^N$ into $\mathscr{F}^{(n-k)k}$ it follows immediately that h_γ^{-1} is continuous. Now, on $S_A(k; p_1, \ldots, p_l) \cap \overline{U}_\gamma$, μ^{-1} is equal to $\overline{\psi}_\gamma^{-1} \circ h_\gamma^{-1} \circ \overline{\phi}'_\gamma$ which shows that μ^{-1} is continuous on $S_A(k; p_1, \ldots, p_l) \cap \overline{U}_\gamma$. Since $\{S_A(k; p_1, \ldots, p_l) \cap \overline{U}_\gamma\}_{\gamma \in \Gamma}$ is an open

cover of $S_A(k; p_1, \ldots, p_l)$ (in the relative topology from $G^k(\mathscr{F}^n)$), this shows that μ^{-1} is continuous. Hence, μ is an embedding of $\mathscr{B}/G$ into $G^k(\mathscr{F}^n)$ with image $S_A(k; p_1, \ldots, p_l)$. Thus, $S_A(k; p_1, \ldots, p_l)$ is a regular submanifold of $G^k(\mathscr{F}^n)$ of dimension $\sum_{i=1}^d q_i(c_i - q_i)$. $\square$

COROLLARY OF PROOF. $S_A(k; p_1, \ldots, p_l)$ is connected.

PROOF. Let $\gamma \in \Gamma(k; p_1, \ldots, p_l)$. Since V_γ is dense in $\mathscr{B}(k; p_1, \ldots, p_l)$, $\overline{V}_\gamma$ is dense in $\mathscr{B}/G$. Since $\overline{\psi}_\gamma$ maps $\overline{V}_\gamma$ homeomorphically onto $\mathscr{F}^N$, $\overline{V}_\gamma$ is connected. Hence, $\mathscr{B}/G$ is connected. $\square$

In §4, we showed that $S_A(k)$ is connected (in the relative topology from $G^k(\mathscr{F}^n)$, $\mathscr{F} = \mathbf{R}$ or $\mathbf{C}$), but it is not generally a submanifold. Theorem 7 shows that if we fix not only the dimension, k, but also the cyclic structure, $(p_1, \ldots, p_l)$, then the set of k-dimensional A-invariant subspaces with this cyclic structure, $S_A(k; p_1, \ldots, p_l)$, is a regular submanifold of $G^k(\mathscr{F}^n)$. Recall that $P_A(k)$ is the set of partitions of k which are compatible with the block structure of A, and $P_A(k)$ contains a smallest element p_* in the natural ordering. By Proposition 5, $S_A(k; p_*)$ is a Grassmannian submanifold of $G^k(\mathscr{F}^n)$. By the Corollary to Lemma 2, if $p \in P_A(k)$, then $S_A(k; p)$ is closed in $G^k(\mathscr{F}^n)$ iff $p = p_*$. Thus, $S_A(k)$ is the set-theoretic disjoint union $\bigsqcup_{p \in P_A(k)} S_A(k; p)$ of submanifolds of $G^k(\mathscr{F}^n)$, only one of which is closed.

REMARK 10. The results in this section have been derived under the assumption that A is nilpotent. In §3, we showed that this is no loss of generality.

6. Schubert varieties. As before, let V be an n-dimensional vector space over the field $\mathscr{F}$ of real or complex numbers. Let $W_1 \subseteq \cdots \subseteq W_k$ be a flag, or nested sequence of linear subspaces of V, and let n_j be the dimension of W_j ($j = 1, \ldots, k$). Let $[W_1, \ldots, W_k] \equiv \{S \in G^k(V): \dim S \cap W_j \geq j, \; j = 1, \ldots, k\}$. The image of $[W_1, \ldots, W_k]$ under the Plücker map p is the intersection of $p(G^k(V))$ with the zero set of a family of homogeneous linear polynomials [6]. In particular, this means that $[W_1, \ldots, W_k]$ is a subvariety of $G^k(V)$. It is known as a Schubert variety. We have seen in §2 that $p(S_A(k))$ is also the intersection of $p(G^k(V))$ with the zero set of a family of homogeneous linear polynomials. (If A has more than one distinct eigenvalue, then strictly speaking, $p(S_A(k))$ is a disjoint union of such objects.) Thus, a natural question to ask is whether $S_A(k)$ is a Schubert variety, or more generally, a union or intersection of Schubert varieties. By analyzing an example in detail, we show that this need not be the case.

Consider the operator $A: \mathbf{R}^4 \to \mathbf{R}^4$ defined by

$$A \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Using the notation introduced in §4, the partition of $n = 4$ corresponding to the block structure of A is $(m_1, m_2) = (2, 2)$. The 2-dimensional A-invariant subspaces, $S_A(2)$, form a compact subvariety of $G^2(\mathbf{R}^4)$. We examime the geometric structure of this subvariety.

There are 2 partitions of "2" which are compatible with the block structure of A, namely (2) and (1,1). Thus $S_A(2)$ is the set-theoretic disjoint union $S_A(2;2) \cup S_A(2;1,1)$. If $\{e_1, e_2, e_3, e_4\}$ is the standard basis for $\mathbf{R}^4$, then $S_A(2;1,1)$ contains only 1 point, $\text{Sp}\{e_2, e_4\}$.

The elements of $S_A(2;2)$ are the cyclic 2-dimensional subspaces. Every such subspace can be expressed in the form $\text{Sp}\{(x_1, x_2, x_3, x_4)(0, x_1, 0, x_3)\}$ where x_1 and x_3 are not both zero. By normalizing, we may assume that $x_1^2 + x_3^2 = 1$, so we can let $x_1 \equiv \cos\theta$ and $x_2 \equiv \sin\theta$. Thus, every element of $S_A(2;2)$ has the form $\text{Sp}\{(\cos\theta, x_2, \sin\theta, x_4), (0, \cos\theta, 0, \sin\theta)\}$. However, there is no loss of generality in requiring that the spanning vectors be mutually orthogonal. This gives a representation of the form $\text{Sp}\{(\cos\theta, -r\sin\theta, \sin\theta, r\cos\theta), (0, \cos\theta, 0, \sin\theta)\}$. Thus we have a surjection $f: S^1 \times \mathbf{R}^1 \to S_A(2;2)$ given by

$$f((\cos\theta, \sin\theta), r) = \text{Sp}\{(\cos\theta, -r\sin\theta, \sin\theta, r\cos\theta), (0, \cos\theta, 0, \sin\theta)\}.$$

It is easy to check that $f((\cos\theta_1, \sin\theta_1), r_1) = f((\cos\theta_2, \sin\theta_2), r_2)$ iff $(\cos\theta_1, \sin\theta_1) = \pm(\cos\theta_2, \sin\theta_2)$ and $r_1 = r_2$. Hence, if we define an equivalence relation $\sim$ on $S^1 \times \mathbf{R}^1$ such that $((\cos\theta_1, \sin\theta_1), r_1) \sim ((\cos\theta_2, \sin\theta_2), r_2)$ iff $(\cos\theta_1, \sin\theta_1) = \pm(\cos\theta_2, \sin\theta_2)$ and $r_1 = r_2$, then the induced mapping $\bar{f}: S^1 \times \mathbf{R}^1/\sim \to S_A(2;2)$ is bijective. It is in fact a homeomorphism. Since $S^1 \times \mathbf{R}^1/\sim$ is homeomorphic to $S^1 \times \mathbf{R}^1$, this shows that $S_A(2;2)$ is a cylinder.

Since $S_A(2)$ is compact, the single point $\text{Sp}\{e_2, e_4\} \in S_A(2;1,1)$ must compactify the cylinder. It is not hard to see that this point compactifies $S^1 \times \mathbf{R}^1$ to give a pinched torus. In particular, $\text{Sp}\{e_2, e_4\}$ is a singular point of the variety $S_A(2)$ and is the only such point.

There are only 4 types of nontrivial Schubert varieties in $G^2(\mathbf{R}^4)$ [4, p. 197]. In codimension 1, there is the Schubert hypersurface consisting of all 2-planes which intersect a given 2-plane. In codimension 2, there are 2 types: the set of all 2-planes which are contained in a given 3-plane, and the set of all 2-planes which contain a given 1-plane. In codimension 3, there is the set of all 2-planes which contain a given 1-plane and are contained in a given 3-plane. The codimension 2 Schubert varieties are both isomorphic to $G^2(\mathbf{R}^3)$ while the codimension 3 Schubert variety is isomorphic to $G^1(\mathbf{R}^2)$. Thus, it is clear that $S_A(2)$ is not a union of Schubert varieties in $G^2(\mathbf{R}^4)$.

The remaining question is whether $S_A(2)$ is the intersection of 2 Schubert hypersurfaces. Every element of $S_A(2)$ intersects $\text{Sp}\{e_2, e_4\}$, so $S_A(2)$ is contained in the Schubert hypersurface determined by $\text{Sp}\{e_2, e_4\}$. Let $M \equiv \text{Sp}\{(m_{11}, m_{21}, m_{31}, m_{41}), (m_{12}, m_{22}, m_{32}, m_{42})\}$ be a 2-plane in $\mathbf{R}^4$. From §5, $S_A(2;2)$ is covered by 2 charts: $(x_3, x_4) \to \text{Sp}\{(1, 0, x_3, x_4), (0, 1, 0, x_3)\}$ and $(x_1, x_2) \to \text{Sp}\{(x_1, x_2, 1, 0), (0, x_1, 0, 1)\}$. If $S_A(2;2)$ is contained in the Schubert hypersurface determined by M, then

$$\det\{(1, 0, x_3, x_4), (0, 1, 0, x_3), (m_{11}, m_{21}, m_{31}, m_{41}), (m_{12}, m_{22}, m_{32}, m_{42})\}$$

must vanish for every x_3, x_4. An elementary calculation shows that this occurs iff $m_{11} = m_{12} = m_{31} = m_{32} = 0$ — i.e. iff $M = \text{Sp}\{e_2, e_4\}$. Hence the only Schubert hypersurface which contains $S_A(2)$ is the one determined by $\text{Sp}\{e_2, e_4\}$. So $S_A(2)$ is

not the intersection of 2 Schubert hypersurfaces. Thus, $S_A(2)$ is not a union or intersection of Schubert varieties in $G^2(\mathbf{R}^4)$.

REMARK 11. In §4 we discussed the natural flag of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_{m_1} = V$ associated with a nilpotent operator A with block structure $(m_1, \ldots, m_r)$. V_i is the kernel of A^i. Let $p = (p_1, \ldots, p_l)$ be a partition of k which is compatible with the block structure of A. Let $(q_1, \ldots, q_d)$ be the conjugate partition of $(p_1, \ldots, p_l)$. If $S \in S_A(k)$, then $S \in S_A(k; p_1, \ldots, p_l)$ iff $\dim S \cap V_1 = q_1$, $\dim S \cap V_2 = q_1 + q_2, \ldots, \dim S \cap V_d = q_1 + \cdots + q_d = k$. Let $\sigma(q_1, \ldots, q_d) \equiv \{ S \in G^k(V) : \dim S \cap V_i \geq q_1 + \cdots + q_i \ (i = 1, \ldots, d) \}$. Then $\sigma(q_1, \ldots, q_d)$ is a Schubert variety, and $S_A(k) \cap \sigma(q_1, \ldots, q_d)$ is the union $\bigcup_{p' \leq p} S_A(k; p'_1, \ldots, p'_{l'})$. In this expression, $p' = (p'_1, \ldots, p'_{l'})$ and the union is taken over every partition p' of k which is compatible with the block structure of A and which is less than or equal to p in the natural ordering on partitions of k. (We use here the fact that conjugation is order-reversing.)

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